STUDY MATERIAL
SOLID MECHANICS UNIT- I SHORT QUESTION AND ANSWERS

What is stress tensor mean?
The **tensor** consists of nine components that completely define the state of **stress** at a point inside a material in the deformed state, placement, or configuration. The **tensor** relates a unit-length direction vector n to the **stress** vector T across an imaginary surface perpendicular to n: The unit vector is dimensionless

What is stress tensor mean?
The **tensor** consists of nine components that completely define the state of **stress** at a point inside a material in the deformed state, placement, or configuration. The **tensor** relates a unit-length direction vector n to the **stress** vector T across an imaginary surface perpendicular to n: The unit vector is dimensionless.

What is stress and strain tensor?
Stress Tensor:- **Stress** is defined as force per unit area. If we take a cube of material and subject it to an arbitrary load we can measure the **stress** on it in various directions   These measurements will form a second rank **tensor**; the **stress tensor**.

Define strain energy
Energy stored in an elastic body under loading. "ligaments and tendons are elastic structures that can store strain energy, like a spring"

Definition of 'plane stress'
Plane stress exists when one of the three principal **stresses** is zero. In very flat or thin objects, the **stresses** are negligible in the smallest dimension so **plane stress** can be said to apply. **Plane stress** is a two-dimensional state of **stress** in which all **stress** is applied in a single **plane**.

What is normal stress?
A **normal stress** is a **stress** that occurs when a member is loaded by an axial force. The value of the **normal** force for any prismatic section is simply the force divided by the cross sectional area. A **normal stress** will occur when a member is placed in tension or compression.

What is major principal stress?
Principal Stresses. It is defined as the normal **stress** calculated at an angle when **shear stress** is considered as zero. The maximum value of normal **stress** is known as **major principal stress** and minimum value of normal **stress** is known as minor **principal stress**.

What is stress in material?
In continuum mechanics, **stress** is a physical quantity that expresses the internal forces that neighboring particles of a continuous **material** exert on each other, while **strain** is the measure of the deformation of the **material** which is not a physical quantity.

What is Mohr's circle used for?

The **Mohr circle** is **used to** find the stress components and i.e., coordinates of any point on the **circle**, acting on any other plane passing through making an angle with the plane. For this, two approaches can be **used**: the double angle, and the Pole or origin of planes.

What is a bending stress?

Bending stress is the normal **stress** that is induced at a point in a body subjected to loads that cause it to bend. When a load is applied perpendicular to the length of a beam (with two supports on each end), **bending** moments are induced in the beam. The bottom fibers of the beam undergo a normal tensile **stress**.

What are the 3 principal stresses?

These **three principal stress** can be found by solving the following cubic equation, This equation will give **three** roots, which will be the **three principal stresses** for the given **three** normal **stresses** (σ_x, σ_y and σ_z) and the **three** shear **stresses** (τ_{xy}, $\tau_{yz}$and τ_{zx}).

What are different types of stresses?

There are **six types of stress**: compression, tension, shear, bending, torsion, and fatigue. Each of these **stresses** affects an object in **different** ways and is caused by the internal forces acting on the object.

Why is the strain tensor symmetric?

It is defined to be **symmetric,** so that it behaves like a **tensor**. ... The stress **tensor**, which is its energy conjugate, is **symmetric**, and hence the skew-**symmetric** part has no contribution towards **strain** energy

Is the stress tensor always symmetric?

The components of the Cauchy **stress tensor** at every point in a material satisfy the equilibrium equations (Cauchy's equations of motion for zero acceleration). Moreover, the principle of conservation of angular momentum implies that the **stress tensor** is **symmetric**."

What is stress tensor in engineering?

The **Stress Tensor**

Stress is defined as force per unit area. If we take a cube of material and subject it to an arbitrary load we can measure the **stress** on it in various directions. These measurements will form a second rank **tensor**; the **stress tensor**.

Define Cauchy's relation

Cauchy's equation is an empirical **relationship** between the refractive index and wavelength of light for a particular transparent material. It is named for the mathematician Augustin-Louis **Cauchy**, who **defined** it in 1836

Define Compatibility

Compatibility conditions are mathematical **conditions** that determine whether a particular deformation will leave a body in a **compatible** state. In the context of infinitesimal strain theory, these **conditions** are equivalent to stating that the displacements in a body can be obtained by integrating the strains.

What is meant by strain compatibility?
In the two-dimensional case, there are three **strain**-displacement relations but only two displacement components. This implies that the **strains** are not independent but are related in some way. The relations between the **strains** are called **compatibility** conditions.

What do you mean by stress function?
The Airy **stress function**: Scalar potential **function** that **can** be used to find the stress. Satisfies equilibrium in the absence of body forces. Only for two-dimensional problems (plane stress/plane strain)

What is being compatible in a relationship?
Love, on the other hand, is a deeper emotion that you feel for another person. ... It also has an emotional and sexual nature unlike **compatibility**, which doesn't always." Basically, **being** in a **compatible relationship** means that you work well together, enjoy each other's company and have a good time

What is compatibility equation?
Compatibility equations are those additional **equations** which can be made considering equilibrium of the structure, to solve statically indeterminate structures

Dr.M.R.RAJAMANICKAM,
ASSOCIATE PROFESSOR,
MECHANICAL ENGINEERING,
ANNAMALAI UNIVERSITY

SOLID MECHANICS UNIT-II

Constitutive equations: Generalized Hooke's Law, Linear elasticity, Material Symmetry

What is meant by constitutive matrix?

In physics and engineering, a **constitutive** equation or **constitutive** relation is a relation between two physical quantities (especially kinetic quantities as related to kinematic quantities) that is specific to a material or substance, and approximates the response of that material to external stimuli.

What is a mechanical constitutive equation?

(**Mechanical** engineering: Mechanics and dynamics) A **constitutive equation** is an**equation** that describes the relationship between two physical quantities, for example between the stress put on a material and the strain produced on it. The **constitutive equation** for most metals is based on Hooke's law.

What is constitutive modeling?

Constitutive modelling is the mathematical description of how materials respond to various loadings. This is the most intensely researched field within solid mechanics because of its complexity and the importance of accurate **constitutive models** for practical engineering problems.

What is compliance tensor?

The **stiffness** and **compliance tensors**

For hyper elastic materials, the stress and strain of a linear **elastic** material are such that one can be derived from a stored energy potential function of the other (also called a strain energy density function)

Is Hooke's law a constitutive equation?

Definition of '**constitutive equation**'

A **constitutive equation** is an **equation** that describes the relationship between two physical quantities, for example between the stress put on a material and the strain produced on it. The **constitutive equation** for most metals is based on **Hooke's law**.

What is compatibility equation?

Compatibility equations are those additional **equations** which can be made considering equilibrium of the structure, to solve statically indeterminate structures. Take the case of a cantilever propped at its free end. ... So, we need 1 extra **compatibility equation**, in addition to the three equilibrium **equations**.

What is monoclinic material?

Monoclinic materials:

As there is a single plane of **material** property symmetry, shear stresses from the planes in which one of the axis is the perpendicular axis of the plane of **material** symmetry (i.e.; 2-3 and 3-1 planes) will contribute only to the shear strains in those planes.

What is transversely isotropic material?

A **transversely isotropic material** is one with physical properties that are symmetric about an axis that is normal to a plane of **isotropy**. This **transverse** plane has infinite planes of symmetry and thus, within this plane, the **material** properties are the same in all directions.

What does Hyper elastic mean?

A **hyper elastic** or green elastic material **is** a type of constitutive model for ideally elastic material for which the stress–strain relationship derives from a strain energy density function.

What is tensor in SOM?

Tensors are referred to by their "rank" which is a description of the **tensor's** dimension. A zero rank **tensor** is a scalar, a first rank **tensor** is a vector; a one-dimensional array of numbers. A third rank **tensor** would look like a three-dimensional matrix; a cube of numbers

What is the difference between orthotropic and anisotropic?

Orthotropic materials are a subset of **anisotropic** materials; their properties depend on the direction in which they are measured. **Orthotropic** materials have three planes/axes of symmetry. An **isotropic** material, in contrast, has the same properties in every direction.

How many independent elastic constants are there for an isotropic material?

There are 81 **independent elastic constants** for generally anisotropic **material** and two for an **isotropic material**. Let us summarize the reduction of **elastic constants** from generally anisotropic to **isotropic material**. For a generally anisotropic **material there** are 81 **independent elastic constants**.

What is strain compatibility method?

A concrete stress block is used with a **strain compatibility method** to predict flexural and axial strengths of concrete-filled tube columns. The accurate stress-**strain** relations of the confined concrete and steel should be used to get an exact solution while using the **strain compatibility method**.

What is compatibility condition?

Compatibility conditions are mathematical **conditions** that determine whether a particular deformation will leave a body in a **compatible** state. In the context of infinitesimal strain theory, these **conditions** are equivalent to stating that the displacements in a body can be obtained by integrating the strains.

Are composites homogeneous?

A **homogeneous** material is one where properties are uniform throughout, i.e. they do not depend on position in body. An isotropic material is one where properties are direction independent. **Composites** are inhomogeneous (or heterogeneous) as well as non-isotropic in nature.

Are composites isotropic or anisotropic?

Anisotropic materials have different material properties in all directions at a point in the body. Bulk materials, such as metals and polymers, are normally treated as **isotropic** materials, while **composites** are treated as **anisotropic**. **Composites** are a subclass of **anisotropic** materials that are classified as orthotropic.

What is isotropic material?

Isotropic material means a **material** having identical values of a property in all directions. Glass and metals are examples of **isotropic materials**. Anisotropic **material's** properties such as Young's Modulus, change with direction along the object. Common examples of anisotropic **materials** are wood and composites.

Chapter 9

CONSTITUTIVE RELATIONS FOR LINEAR ELASTIC SOLIDS

Figure 9.1: Hooke memorial window, St. Helen's, Bishopsgate, City of London

9.1 Mechanical Constitutive Equations

Recall that in Chapters 2, 3 and 8 we briefly introduced the concept of a constitutive equation, which generally relates kinetic variables to kinematic variables in the application of interest. With respect to the application of the analysis of mechanical deformations in solids, the kinetic variable is the stress tensor, $\boldsymbol{\sigma}$, whereas the kinematic variables are the displacements u_x, u_y, u_z, and the strain tensor, $\boldsymbol{\varepsilon}$ which includes derivatives (sometimes called gradients) of the displacements. Since it is generally observed that rigid body displacements do not induce stresses, the displacement field u_x, u_y, u_z, will not enter into a mechanical constitutive equation. Thus, the constitutive equations will in general relate stress, $\boldsymbol{\sigma}$, to strains, $\boldsymbol{\varepsilon}$, and temperature T. In 1660, Robert Hooke observed that for a broad class of solid materials called linear elastic (or Hookean), this relationship may be described by a linear relationship. Hooke originally considered the test of a uniaxial body with a force (stress) applied only in one direction and measured the corresponding elongation (strain) to obtain:

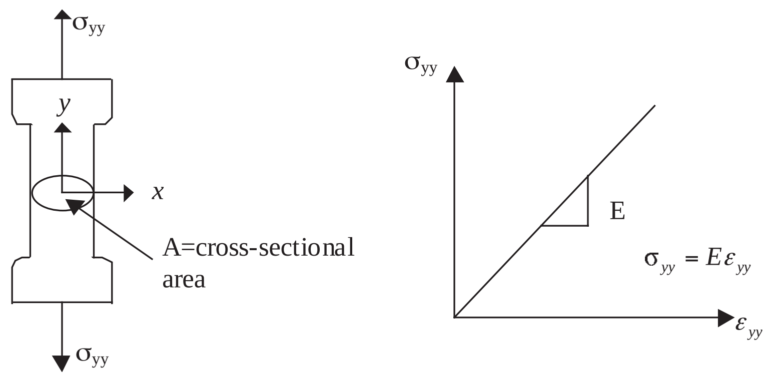


Figure 9.2: Stress-Strain Curve for Linear Elastic Material

For a general three-dimensional state of stress, there are 6 independent stresses and 6 independent strains; therefore, the linear relationship between stress and strain can be written in matrix form as:

$$\boldsymbol{\sigma} = [\mathbf{C}]\,\boldsymbol{\varepsilon} \tag{9.1}$$

where $[\mathbf{C}]$ is a 6×6 matrix of elastic constants that must be determined from experiments. In expanded form, these 6 equations become:

$$\begin{aligned}
\sigma_{xx} &= C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} + C_{14}\varepsilon_{yz} + C_{15}\varepsilon_{zx} + C_{16}\varepsilon_{xy} \\
\sigma_{yy} &= C_{21}\varepsilon_{xx} + C_{22}\varepsilon_{yy} + C_{23}\varepsilon_{zz} + C_{24}\varepsilon_{yz} + C_{25}\varepsilon_{zx} + C_{26}\varepsilon_{xy} \\
\sigma_{zz} &= C_{31}\varepsilon_{xx} + C_{32}\varepsilon_{yy} + C_{33}\varepsilon_{zz} + C_{34}\varepsilon_{yz} + C_{35}\varepsilon_{zx} + C_{36}\varepsilon_{xy} \\
\sigma_{yz} &= C_{41}\varepsilon_{xx} + C_{42}\varepsilon_{yy} + C_{43}\varepsilon_{zz} + C_{44}\varepsilon_{yz} + C_{45}\varepsilon_{zx} + C_{46}\varepsilon_{xy} \\
\sigma_{zx} &= C_{51}\varepsilon_{xx} + C_{52}\varepsilon_{yy} + C_{53}\varepsilon_{zz} + C_{54}\varepsilon_{yz} + C_{55}\varepsilon_{zx} + C_{56}\varepsilon_{xy} \\
\sigma_{xy} &= C_{61}\varepsilon_{xx} + C_{62}\varepsilon_{yy} + C_{63}\varepsilon_{zz} + C_{64}\varepsilon_{yz} + C_{65}\varepsilon_{zx} + C_{66}\varepsilon_{xy}
\end{aligned} \tag{9.2}$$

It is interesting to note that Robert Hooke first proposed the above "law" publicly in an anagram at Hampton Court (1676) given by the group of letters:

ceiiinosssttuv.

In 1678 he explained the anagram to be

"Ut tensio sic vis,"

which is Latin meaning "as the tension so the displacement" or in English "the force is proportional to the displacement." Students may recall that during this time period, science and scientific writing was criticized and hence Hooke thought it necessary to discretely disclose his scientific finding with an anagram.

Note that in the previous chapters stress and strain were represented as (3×3) matrices. It is convenient, however, here to represent them as (6×1) column vectors since they have only 6 independent components (stress due to conservation of angular momentum and strain by its definition). We then write

$$\{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix}, \quad \{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{Bmatrix}$$

By adopting this representation for $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, their linear relationship (9.2) can be easily written in matrix form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{Bmatrix} \tag{9.3}$$

- If a material is *homogeneous* then the constants (C_{ij}, $i = 1, ..., 6$ and $j = 1, ..., 6$) are all independent of x, y, z for any time, t.

- If a material is *isotropic*, then for a given material point, C_{ij} are independent of the orientation of the coordinate system (i.e., the material properties are the same in all directions).

- If a material is *orthotropic*, then for a given material point, C_{ij} can be defined in terms of properties in three orthogonal coordinate directions.

- If a material is *anisotropic*, then for a given material point, C_{ij} are different for all orientations of the coordinate system.

In order to determine the material constants in equation (9.3), consider a **uniaxial tensile test** using a test specimen of *linear elastic isotropic material* with cross-sectional area A and subjected to a uniaxilly applied load F in the axial (y) direction as shown below. The cross-section may be any shape but generally a rectangular or cylindrical shape is chosen. For a rectangular specimen, assume a width W and thickness t so that the cross-sectional area is $A = Wt$. Assume a small *gauge length* of L for which the axial deformation will be measured during the load application.

During the **uniaxial tensile test**, we observe that the gauge length changes from L to L^* and the gauge width *decreases* from W to W^*. We also observe a decrease (contraction) in the z dimension. We further observe no change in angular orientation of the vertical or horizontal elements and conclude that for uniaxial loading, no shear strains are produced. This leads us to postulate the following strain state: $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz} \neq 0$, $\varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{xz} = 0$. The axial stress and strain in the axial (y) direction are defined to be

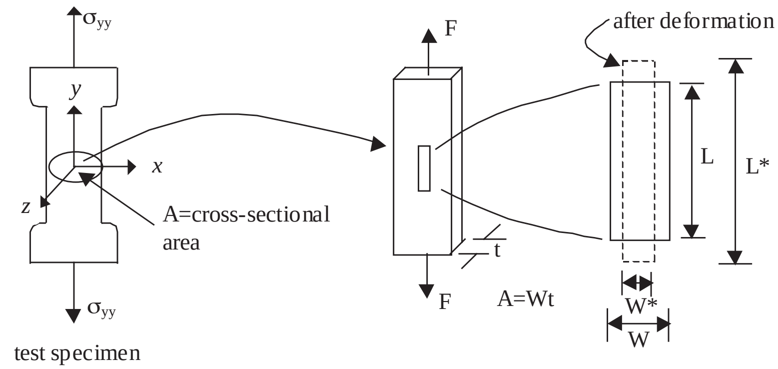


Figure 9.3: Experimental Measurement of Axial and Transverse Deformation

$$\sigma_{yy} = \frac{F}{A}$$
$$\varepsilon_{yy} = \frac{\Delta L}{L} = \frac{(L^* - L)}{L}$$

The strain the in the transverse (x) direction due to the axial load is

$$\varepsilon_{xx} = \frac{\Delta W}{W} = \frac{(W^* - W)}{W}$$

If we plot *axial stress vs. axial strain* and *transverse strain vs. axial strain*, we obtain the following two plots:

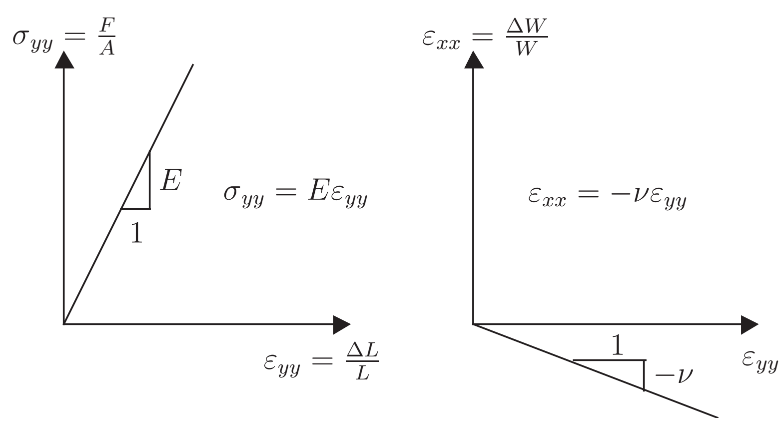


Figure 9.4: Experimental Results for Axial Stress vs. Axial Strain & Transverse Strain vs. Axial Strain

From these two plots, we can write $\sigma_{yy} = E\varepsilon_{yy}$ and $\varepsilon_{xx} = -\nu\varepsilon_{yy}$ for the uniaxial tension test. Consequently, we may define the following two material constants from this single uniaxial test:

- E = slope of the uniaxial σ_{yy} vs. ε_{yy} curve = *a material constant called Young's modulus*

- $\nu = -\frac{\varepsilon_{xx}}{\varepsilon_{yy}}$ = negative ratio of the strain normal to the direction of loading over the strain in the loading direction = *a material constant called Poisson's ratio*

If the transverse strain were measured in the z direction, we would find the same ratio for transverse to axial strain: $\nu = -\frac{\varepsilon_{zz}}{\varepsilon_{yy}}$.

Combining these equations, we can write the two transverse strains entirely in terms of the axial stress σ_{yy}: $\varepsilon_{xx} = -\nu\varepsilon_{yy} = -\nu\left(\frac{\sigma_{yy}}{E}\right)$ and $\varepsilon_{zz} = -\nu\varepsilon_{yy} = -\nu\left(\frac{\sigma_{yy}}{E}\right)$.

In order to obtain a complete description of three-dimensional constitutive behavior, consider a test where we apply *normal tractions (stresses) in the x, y and z* directions simultaneously and measure the strain *only* in the x direction. For a linear material response, we may use the principle of linear superposition and consider three separate cases as shown below:

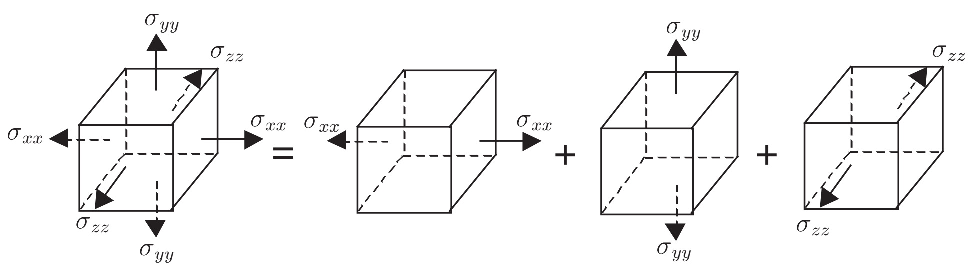


Figure 9.5: Experimental Test with all Components of Normal Stresses Applied

$$
\begin{aligned}
\varepsilon_{xx} &= \text{normal strain in } x \text{ direction due to } \sigma_{xx} \\
&+ \text{normal strain in } x \text{ direction due to } \sigma_{yy} \\
&+ \text{normal strain in } x \text{ direction due to } \sigma_{zz} \\
&= \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}
\end{aligned}
$$

or

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \tag{9.4}$$

The stress in the x direction increases the strain in the x direction while the transverse stresses causes a contraction (decrease in ε_{xx}).

Doing similar experiments in the y and z directions gives:

$$
\begin{aligned}
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right]
\end{aligned} \tag{9.5}
$$

Experiments with shear tractions will show that a shear stress σ_{xy} in the x-y plane produces only shear strain ε_{xy} in the x-y plane for a state of pure shear loading (i.e., no normal strain is observed so that the shear strain is uncoupled from the normal strain).[1] Thus, we obtain the following

[1] Keep in mind that even for the case of pure shear, if one calculates shear stresses (or strains) at some angle θ from the x-axis (Mohr's circle), one may obtain non-zero normal stresses (or strains) for the off-axis planes.

experimental observations for the shear strains:

$$
\begin{aligned}
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\varepsilon_{xz} &= \frac{1+\nu}{E}\sigma_{xz} \\
\varepsilon_{yz} &= \frac{1+\nu}{E}\sigma_{yz}
\end{aligned} \tag{9.6}
$$

Combining equations (9.4), (9.5) and (9.6), Hooke's law for a linear elastic isotropic solid with a three-dimensional stress state becomes:

Hooke's Law for a Linear Elastic Isotropic Solid

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\varepsilon_{xz} &= \frac{1+\nu}{E}\sigma_{xz} \\
\varepsilon_{yz} &= \frac{1+\nu}{E}\sigma_{yz}
\end{aligned} \tag{9.7}
$$

where E = Young's modulus and ν = Poisson's ratio.

It should be noted that in materials that undergo permanent deformation, the above model is not accurate (such as metals beyond their yield point, or polymers that flow). A typical uniaxial stress-strain curve for a ductile metal is shown below:

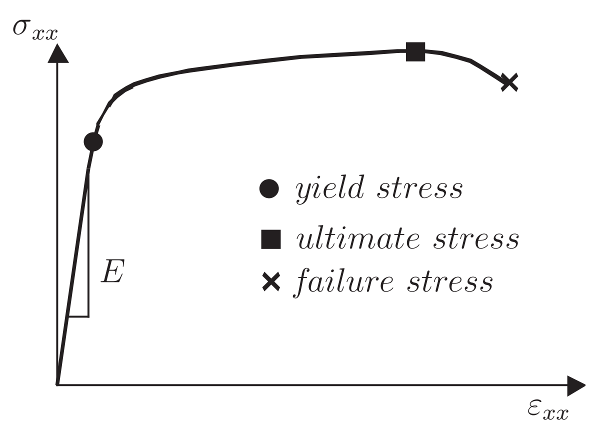


Figure 9.6: Typical Stress-Strain Curve for Ductile Metal

An algebraic inversion of the strain-stress relationship (9.7) provides the following relationship of stress in terms of strain:

$$\{\boldsymbol{\sigma}\} = \frac{E}{1+\nu}\begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\left\{\begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{array}\right\} \tag{9.8}$$

or,

Hooke's Law for a Linear Elastic Isotropic Solid

$$\{\boldsymbol{\sigma}\} = \left\{\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{array}\right\} = \left\{\begin{array}{c} \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}\right] \\ \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} - \nu\varepsilon_{zz}\right] \\ \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}\right] \\ \frac{E}{1+\nu}\varepsilon_{xy} \\ \frac{E}{1+\nu}\varepsilon_{xz} \\ \frac{E}{1+\nu}\varepsilon_{yz} \end{array}\right\} \tag{9.9}$$

where $E =$ Young's modulus and $\nu =$ Poisson's ratio.

The term $\frac{E}{(1+\nu)} \equiv 2G$ defines a *shear modulus, G,* relating shear strain and shear stress (similar to Young's modulus, E, for extensional strain). Thus, the shear modulus is given by:

$$G = \frac{E}{2(1+\nu)} \tag{9.10}$$

Note that the shear strain ε_{xy} is related to engineering shear strain γ_{xy} by $\gamma_{xy} = 2\varepsilon_{xy} = 2(\frac{1+\nu}{E})\sigma_{xy} = \frac{\sigma_{xy}}{G}$ so that $\sigma_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy}$.

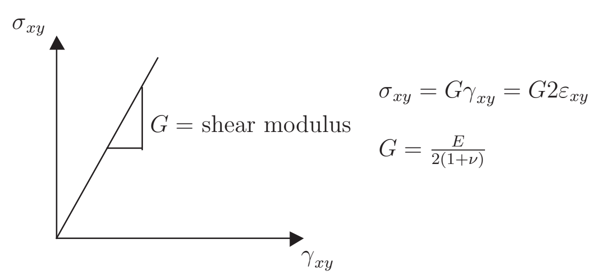


Figure 9.7: Experimental Results for Shear Stress vs. Engineering Shear Strain

Note that G is defined in terms of E and ν and consequently G is not a new material property. Thus, for a *homogeneous linear elastic isotropic solid*, we conclude that only two material properties (Young's modulus, E, and Poisson's ratio, ν) are required to completely define the three-dimensional constitutive behavior.

The stress-strain equations may also be written in terms of shear modulus to obtain:

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx}+\nu\varepsilon_{yy}+\nu\varepsilon_{zz}\right] = \frac{2G}{1-2\nu}\left[(1-\nu)\,\varepsilon_{xx}+\nu\varepsilon_{yy}+\nu\varepsilon_{zz}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx}+(1-\nu)\varepsilon_{yy}-\nu\varepsilon_{zz}\right] = \frac{2G}{1-2\nu}\left[\nu\varepsilon_{xx}+(1-\nu)\,\varepsilon_{yy}+\nu\varepsilon_{zz}\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx}+\nu\varepsilon_{yy}+(1-\nu)\varepsilon_{zz}\right] = \frac{2G}{1-2\nu}\left[\nu\varepsilon_{xx}+\nu\varepsilon_{yy}+(1-\nu)\,\varepsilon_{zz}\right] \\
\sigma_{yz} &= \frac{E}{1+\nu}\varepsilon_{yz} = 2G\varepsilon_{yz} \\
\sigma_{zx} &= \frac{E}{1+\nu}\varepsilon_{zx} = 2G\varepsilon_{zx} \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} = 2G\varepsilon_{xy}
\end{aligned} \tag{9.11}$$

Side Note: If the definition of ε_{xx} and ε_{zz} from $\nu = -\frac{\varepsilon_{xx}}{\varepsilon_{yy}} = -\frac{\varepsilon_{zz}}{\varepsilon_{yy}}$ is substituted into equation (9.9), we obtain for the uniaxial bar extension experiment described previously:

$$\begin{aligned}
&\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)(-\nu\varepsilon_{yy})+\nu\varepsilon_{yy}+\nu(-\nu)\varepsilon_{yy}\right] = 0 \\
&\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[-\nu^2\varepsilon_{yy}+(1-\nu)\varepsilon_{yy}-\nu^2\varepsilon_{yy}\right] = E\varepsilon_{yy} \\
&\sigma_{zz} = 0 \\
&\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0
\end{aligned}$$

This result is consistent with all observations made regarding the nature of stress for the uniaxial test with an applied stress of σ_{yy}.

9.2 Constitutive Equations with Thermal Strain

Experimentally, we observe for a linear isotropic metal that a temperature increase, ΔT, produces a uniform expansion but no shear and the expansion is proportional to a material constant α (coefficient of thermal expansion). The *additional* strain due to heating is thus: $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \alpha\Delta T$. Thus, the constitutive equation for a linear elastic isotropic solid (9.7) may be modified by the addition of the thermal strain to the normal strain components:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx}-\nu(\sigma_{yy}+\sigma_{zz})]+\alpha\Delta T \\
\varepsilon_{yy} &= \frac{1}{E}[\sigma_{yy}-\nu(\sigma_{xx}+\sigma_{zz})]+\alpha\Delta T \\
\varepsilon_{zz} &= \frac{1}{E}[\sigma_{zz}-\nu(\sigma_{xx}+\sigma_{yy})]+\alpha\Delta T \\
\varepsilon_{xy} &= (\frac{1+\nu}{E})\sigma_{xy} \\
\varepsilon_{xz} &= (\frac{1+\nu}{E})\sigma_{xz} \\
\varepsilon_{yz} &= (\frac{1+\nu}{E})\sigma_{yz}
\end{aligned} \tag{9.12}$$

These equations can be inverted to obtain stress in terms of strain:

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{xx}+\nu\varepsilon_{yy}+\nu\varepsilon_{zz}] - \frac{E\alpha\Delta T}{(1-2\nu)} \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx}+(1-\nu)\varepsilon_{yy}+\nu\varepsilon_{zz}] - \frac{E\alpha\Delta T}{(1-2\nu)} \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx}+\nu\varepsilon_{yy}+(1-\nu)\varepsilon_{zz}] - \frac{E\alpha\Delta T}{(1-2\nu)} \\
\sigma_{xy} &= \frac{E}{2(1+\nu)}\varepsilon_{xy} \\
\sigma_{xz} &= \frac{E}{2(1+\nu)}\varepsilon_{xz} \\
\sigma_{yz} &= \frac{E}{2(1+\nu)}\varepsilon_{yz}
\end{aligned} \tag{9.13}$$

In the above, $\Delta T = \Delta T(x, y, z)$ and represents the increase in temperature from a "reference" temperature where the thermal strain is zero. It should be noted that the first term in the extensional strain terms above (the [ ] term) is due to elastic behavior of the material (i.e., it has Young's modulus in it). The second part is due to thermal strain. We can separate the total strain into elastic and thermal strains:

$$\begin{aligned}
\varepsilon_{xx}^{total} &= \varepsilon_{xx}^{elastic} + \varepsilon^{thermal} \\
\varepsilon_{yy}^{total} &= \varepsilon_{yy}^{elastic} + \varepsilon^{thermal} \\
\varepsilon_{zz}^{total} &= \varepsilon_{zz}^{elastic} + \varepsilon^{thermal}
\end{aligned} \tag{9.14}$$

The elastic (also called mechanical) and thermal terms are given by:

$$\begin{aligned}
\varepsilon_{xx}^{elastic} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})] \\
\varepsilon_{yy}^{elastic} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx}+\sigma_{zz})] \\
\varepsilon_{zz}^{elastic} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx}+\sigma_{yy})] \\
\varepsilon^{thermal} &= \alpha\Delta T
\end{aligned} \tag{9.15}$$

The terms ε_{xx}^{total}, ε_{yy}^{total}, ε_{zz}^{total} represent the total strain as measured or observed, and are thus equal to their deformation gradient definitions, i.e., for small strain,

$$\begin{aligned}
\varepsilon_{xx}^{total} &= \varepsilon_{xx} = \frac{\partial u_x}{\partial x} \\
\varepsilon_{yy}^{total} &= \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \\
\varepsilon_{zz}^{total} &= \varepsilon_{zz} = \frac{\partial u_z}{\partial z}
\end{aligned} \tag{9.16}$$

We state once again that shear strains have no thermal component for an isotropic material. Examples of problems involving thermal strain will be considered in Chapter 10.

Some typical values of material properties for isotropic metals are provided in the table below. Note that the values of E (Young's modulus) are typically in the million psi or GPa range for engineering materials, while the values of ν are between zero and 0.5 ($0 < \nu < 0.5$). The yield strength represents the stress level at which the metal yields (becomes inelastic). For ductile metals, the ultimate tensile strength is typically 10 to 50% higher than the yield strength. It should be noted that material properties for commonly used metals must satisfy specifications established by regulatory agencies. *Values in Tables* 9.1 *and* 9.2 *that are not provided are unknown for purposes of presentation herein and they should not to be interpreted as zero. The reader may with to consult other sources for the omitted values.*

For non-metals, properties may vary significantly depending upon many variables (for example, wood has a Young's modulus varying from 0.1×10^6 psi to 2×10^6 psi depending upon the tree species and direction of wood grain; the modulus of concrete will depend on the concrete/aggregate ratio and curing process). Examples of non-metals commonly used are concrete (ultimate compressive strength of 5 ksi but zero tensile strength; with an elastic modulus in compression of 3×10^6 psi) and Douglas fir (parallel to grain, ultimate compressive strength of 7 ksi; with an elastic modulus of 1.6×10^6 psi).

Material	Density $\left(\frac{\text{lb}}{\text{in}^3}\right)$	Young's Modulus (10^6 psi)	Poisson's Ratio	Yield Strength (ksi)		Ultimate Tensile Strength (ksi)	Coefficient of Thermal Expansion $\left(\frac{10^{-6}}{^\circ\text{F}}\right)$
				Tension	Shear		
Aluminum							
2024-T4	0.100	10.5	0.33	40		62	12.9 (200 °F)
6061-T6	0.098	9.9	0.33	36	21	42	13.0 (70-200 °F)
Steel							
Structural (A36)	0.284	30.0	0.29	36	21	65	6.5
AISI 1025	0.284	29.0	0.32	36		55	6.8 (70-200 °F)
5Cr-Mo-V	0.281	30.0	0.36	200		240	7.1 (80-800 °F)
Copper							
G3-Heat Treated	0.272	16.0		60		110	9.0 (70-570 °F)
Titanium							
Ti-5Al-2.5Sn	0.162	15.5		110		115	5.2 (200-400 °F)
Ti-6M-4V	0.160	16.0	0.34	120	72	130	4.6 (200-400 °F)

Table 9.1: Structural Material Properties for Selected Metals (US Customary Units)

Example 9-1

Given:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} -y\frac{M_z}{I_{zz}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{where } M_z \text{ and } I_{zz} \text{ are constants}$$

Required:

(a) Verify that the stress tensor satisfies the Conservation of Linear Momentum.

(b) Determine the components of the infinitesimal strain tensor, $\boldsymbol{\varepsilon}$.

(c) Determine the components of the displacement, u_x, u_y, and u_z.

(d) Describe the displacement and physical problem described by these equations. Use as reference the figure below.

Material	Density $\left(\frac{\text{Mg}}{\text{m}^3}\right)$	Young's Modulus (GPa)	Poisson's Ratio	Yield Strength (MPa)		Ultimate Tensile Strength (MPa)	Coefficient of Thermal Expansion $\left(\frac{10^{-6}}{{}^\circ\text{C}}\right)$
				Tension	Shear		
Aluminum							
2024-T6	2.79	73.1	0.35	414		469	23
6061-T6	2.71	68.9	0.33	245	145	290	24
Steel							
Structural A36	7.85	207	0.29	248	145	445	12
Stainless 304	7.86	193	0.27	207		517	17
Copper Alloy							
Bronze C86100	8.83	103	0.34	345		655	17
Titanium							
Ti-6Al-4V	4.43	120	0.36	924		1,000	9.4
Ti-6M-4V	4.34	110	0.34	827	495	895	8.3

Table 9.2: Structural Material Properties for Selected Metals (SI Units)

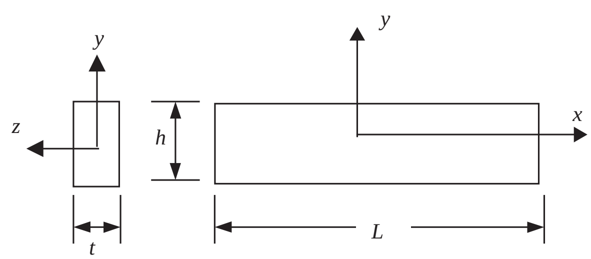


Figure 9.8:

(a) x-component of linear momentum:

$$x \rightarrow \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho g_x = 0$$

$$\frac{\partial \left(-y\frac{M_z}{I_{zz}}\right)}{\partial x} = 0$$

- Stress tensor satisfies the Conservation of Linear Momentum

(b) The strains are given by:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}, \quad \varepsilon_{yy} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx}, \quad \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

$$\varepsilon_{xx} = -y\frac{M_z}{I_{zz}E}, \quad \varepsilon_{yy} = \frac{\nu y M_z}{EI_{zz}}, \quad \varepsilon_{zz} = \frac{\nu y M_z}{EI_{zz}}$$

(c) Integrate the displacement equations and apply boundary conditions:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -y\frac{M_z}{EI_{zz}} \quad \rightarrow \quad u_x = -y\frac{M_z}{EI_{zz}}x + C_1$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{\nu y M_z}{EI_{zz}} \quad \rightarrow \quad u_y = \frac{\nu\left(\frac{y^2}{2}\right)M_z}{EI_{zz}} + C_2$$
$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{\nu y M_z}{EI_{zz}} \quad \rightarrow \quad u_z = \frac{\nu y M_z}{EI_{zz}}z + C_3$$

$$u_x|_{x=0} = 0, \qquad C_1 = 0$$
$$u_y|_{y=0} = 0, \qquad C_2 = 0$$
$$u_z|_{z=0} = 0, \qquad C_3 = 0$$

$$u_x = -yx\frac{M_z}{I_{zz}}$$
$$u_y = \frac{\nu v}{2E}y^2\frac{M_z}{I_{zz}}$$
$$u_z = \frac{\nu}{E}yz\frac{M_z}{I_{zz}}$$

(d) The displacement in the x-direction is negative (shortening) when y is positive due to the negative u_x term. If y is negative the displacement in the x-direction is positive (expanding). In the y-direction and z-direction the displacement is expanding when y is greater than zero and vice versa. Displacement in the y-direction changes with respect to y^2, and in the z-direction it changes with respect to y.

Deep Thought

Ut tensio sic vis!
Ut tensio sic vis!!
Ut tensio sic vis!!!

9.3 Questions

9.1 Which conservation laws are especially useful for describing stresses and strains? How are stress and strain related?

9.2 Write the equations that result from an inversion of the stress-strain relationship.

9.3 Describe in your own words the meanings of the state of plane stress and the state of plane strain?

9.4 Describe the two types of problems which when solved using the theory of plane elasticity provide exact solutions.

9.5 Consider small shear strain for a moment. It is often given in terms of an angle. Explain why this is done.

9.6 What is a constitutive relation? Write down the general constitutive relation in terms of Cauchy stress and strain.

9.7 For an elastic, isotropic solid material, how many constants are required to define the constitutive relations? Name these and define their meaning.

9.4 Problems

9.8 Structural steel is subjected to the deformation defined by $u_x(x,y,z) = 0.002x$, $u_y(x,y,z) = 0$, $u_z(x,y,z) = 0$ (displacements in inches). Determine the following in US units:

a) Infinitesimal strain tensor.
b) Stress tensor.
c) Draw Mohr's Circle for the given state of stress.
d) Principal Stresses and Strains.

9.9 Repeat steps a and b in 9.8 for $u_x(x,y,z) = 0.002x^2 + 0.001x$, $u_y(x,y,z) = 0.002xy$, $u_z(x,y,z) = 0.001z^2$.

9.10 *GIVEN*: A Hookean material with $E = 10 \times 10^6$ psi and $\nu = 0.5$ experiences the following deformation: $u_x(x,y,z) = 0$, $u_y(x,y,z) = 0.004x$, $u_z(x,y,z) = 0$

REQUIRED:

a) Sketch u_x versus x, u_y versus y, and u_z versus z, and calculate $\frac{\partial u_x}{\partial x}$, $\frac{\partial u_y}{\partial y}$, $\frac{\partial u_z}{\partial z}$.
b) Calculate the infinitesimal strain tensor.
c) Calculate the stress tensor.

9.11 *GIVEN*: $\nu = 0.25$ and $E = 2.0 \times 10^{10}$ Pa, and strain tensors as follows

$$(1)\begin{bmatrix} 0.002 & 0.004 & 0 \\ 0.004 & 0.003 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (2)\begin{bmatrix} 0 & 0.005 & 0 \\ 0.005 & 0.04 & 0 \\ 0 & 0 & 0.006 \end{bmatrix}$$

REQUIRED:

(1) Calculate the stress tensors;
(2) How much is the relative volume change (the dilatation) for this deformation, and compare the results obtained by using both finite strain formula and the small strain formula.

9.12 *GIVEN*: $\nu = 0.33$ and $E = 15.0 \times 10^3$ MPa, and stress tensors as follows

$$(1)\begin{bmatrix} 10\text{ MPa} & 4\text{ MPa} & 0 \\ 4\text{ MPa} & 30\text{ MPa} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2)\begin{bmatrix} 20\text{ MPa} & 50\text{ MPa} & 0 \\ 50\text{ MPa} & 0 & 0 \\ 0 & 0 & 6\text{ MPa} \end{bmatrix}$$

REQUIRED: Calculate the strain tensors.

9.13 *GIVEN*: $u_x = z10^{-4}$, $u_z = x10^{-4}$, and $u_y = 0$, and material constants $E = 2.6 \times 10^{10}$, and $\nu = 0.3$.

(1) Compute infinitesimal strain tensor.

(2) Compute the corresponding stress tensor.

(3) What are the principal stresses and principal strains?

(4) Are the principal stresses and strains acting in the same directions?

9.14 A steel plate lies flat in the x-y plane and has dimensions 20 cm × 40 cm. If the plate is uniformly heated throughout at 1000 °C and the thermal expansion coefficient is given by $\alpha = 11 \times 10^{-6} \frac{\text{m}}{(\text{m - }^\circ\text{C})}$, calculate the new dimensions of the plate due to thermal expansion.

9.15 A thin rectangular sheet of linearly elastic material has an x-y coordinate system located at its lower left corner. The body extends 15 in. in the x direction and 8 in. in the y direction. The material is isotropic with an $E = 35,000,000$ psi and $\nu = 0.33$. A plane stress condition has been created by forces acting along the edges of the body with a displacement field of:

$$\begin{aligned} u_x &= 1.44 \times 10^{-8} x^2 y \\ u_y &= -1.44 \times 10^{-8} x y^2 \end{aligned}$$

Write expressions for the surface force normal to and for the tangential surface force along the upper 15 in. boundary as functions of x. Write expressions for the surface force normal to and for the tangential surface force along the right 8 in. boundary as functions of y. Draw the distribution of normal surface force along these two boundaries on a sketch of the body.

9.16 Use web resources to determine the following material properties. Provide the URL (http address) that you used.

(a) Yield strength in shear of 2024-T4 and 2014-T6 aluminum.

(b) Poisson's ratio and yield strength in shear for Ti-5Al-2.5Sn.

(c) All of the table values as presented in Table 9.1 for 4130 heat treated alloy steel.

(d) All of the table values as presented in Table 9.1 for balsa wood.

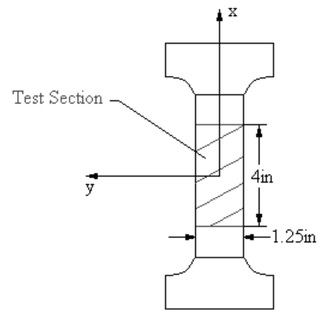
9.17 *GIVEN*: The isothermal (no temperature gradient) uniaxial bar specimen of 2024-T4 Aluminum (isotropic) shown below:

The axial displacements are measured to be:

$$\begin{aligned} u_x &= -0.02x \text{ in} \\ u_y &= 0.000125x - 0.0005 \text{ in} \qquad x \text{ in inches!!} \end{aligned}$$

REQUIRED:

1. Sketch the deformed configuration of the test section boundary (using the displacements given above).
2. Calculate the infinitesimal strain tensor for the test section.



Problem 9.17

3. Calculate the stress tensor for the test section.

9.18 *GIVEN*: A linear isotropic ThermoElastic plate of Stainless 304 is subjected to a uniform temperature change of ΔT and is assumed to be in a state of stress as shown below. At equilibrium, the ΔT is known.

$$\boldsymbol{\sigma} = \begin{bmatrix} -125 & -50 & 0 \\ -50 & -100 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa}$$

REQUIRED:

a) Calculate the infinitesimal strain tensor when $\Delta T = 0$ °C. (review equations 9.12 and 9.13 in the notes)

b) Calculate the infinitesimal strain tensors for the two cases: when $\Delta T = 100$ °C, and when $\Delta T = 25$ °C.

c) Find the temperature change ΔT necessary to produce zero strain.

9.19 *GIVEN*: Consider the state of stress called *plane stress* in which non-zero stresses exist in only one plane.

REQUIRED:

a) For a state of plane stress in the x-y plane, show that the constitutive equations for an elastic isotropic material (isothermal case) reduce to the following. Hint: start with the constitutive equations for the general 3-D elastic, isotropic case and reduce to plane stress; see equation 10.6): SHOW ALL STEPS.

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1-\nu^2)}[\varepsilon_{xx} + \nu\varepsilon_{yy}] \\ \sigma_{yy} &= \frac{E}{(1-\nu^2)}[\nu\varepsilon_{xx} + \varepsilon_{yy}] \\ \sigma_{xy} &= \frac{E}{(1+\nu)}\varepsilon_{xy} \end{aligned}$$

b) Starting with the above relations, show that the strains for plane stress in the x-y plane become those shown below (see equation 10.7). You must *show all steps* necessary to obtain the relations below.

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu\sigma_{yy}] \\
\varepsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu\sigma_{xx}] \\
\varepsilon_{xy} &= (\frac{1+\nu}{E})\sigma_{xy} \\
\varepsilon_{zz} &= -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})
\end{aligned}
$$

You may use Scientific Workplace to do the matrix algebra.

9.20 *GIVEN*: Given that for a general orthotropic elastic material there are 12 unique coeffecients such that:

$$
[D] = \begin{bmatrix}
\frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{11}} & -\frac{\nu_{13}}{E_{11}} & 0 & 0 & 0 \\
-\frac{\nu_{21}}{E_{22}} & \frac{1}{E_{22}} & -\frac{\nu_{23}}{E_{22}} & 0 & 0 & 0 \\
-\frac{\nu_{31}}{E_{33}} & -\frac{\nu_{32}}{E_{33}} & \frac{1}{E_{33}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\mu_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\mu_{31}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_{12}}
\end{bmatrix}
$$

The constitutive equation for this form would then be:

$$\{\varepsilon\} = [D]\{\sigma\}$$

where the stress have the following values

$$
\{\sigma\} = \left\{ \begin{array}{c}
\sigma_{xx} = 5 \text{ ksi} \\
\sigma_{yy} = 10 \text{ ksi} \\
\sigma_{zz} = 20 \text{ ksi} \\
\sigma_{yz} = 0 \text{ ksi} \\
\sigma_{zx} = 0 \text{ ksi} \\
\sigma_{xy} = 7.5 \text{ ksi}
\end{array} \right\}; \qquad
\{\varepsilon\} = \left\{ \begin{array}{c}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{yz} \\
\varepsilon_{zx} \\
\varepsilon_{xy}
\end{array} \right\}
$$

REQUIRED:

a) Write the stress tensor in its more common form (i.e., as a tensor or matrix). Does this constitute generalized plane stress? Why or why not?

   Recall that generalized plane stress is a requirement for Mohr's Circle

b) Suppose that the 12 material coefficients have the following values:

$$
\begin{aligned}
E_{11} &= 10^6 \text{ psi} \\
E_{22} &= 3 \times 10^7 \text{ psi} \\
E_{33} &= 0.2 \times 10^6 \text{ psi}
\end{aligned}
$$

$$
\begin{aligned}
\nu_{12} &= 0.2 \\
\nu_{13} &= 0.25 \\
\nu_{21} &= 0.33 \\
\nu_{23} &= 0.43 \\
\nu_{31} &= 0.05 \\
\nu_{32} &= 0.06
\end{aligned}
$$

$$
\begin{aligned}
\mu_{23} &= 10^4 \text{ psi} \\
\mu_{31} &= 2 \times 10^4 \text{ psi} \\
\mu_{12} &= 3 \times 10^4 \text{ psi}
\end{aligned}
$$

Calculate the infinitesimal strain tensor.

c) Write the strain tensor in its more common form. Does this constitute generalized plane strain? Why or why not?

SOLID MECHANICS SHORT QUESTIONS AND ANSWERS UNIT – III

1.) Definition of 'plane stress'

Plane stress exists when one of the three principal **stresses** is zero. In very flat or thin objects, the **stresses** are negligible in the smallest dimension so **plane stress** can be said to apply. **Plane stress** is a two-dimensional state of **stress** in which all **stress** is applied in a single **plane**

2.) What is plane shear stress?

Shear stress considering the specific **plane** is called in **plane shear stress** and other two **stresses** are out-**plane shear stress**. This type of **stress** generally found in thin cylindrical closed pressure vessel where max

3.) What is meant by principal stress?

Principal Stresses. It is **defined** as the normal **stress** calculated at an angle when **shear stress** is considered as zero. The normal **stress** can be obtained for maximum and minimum values.

4.) What is the difference between von Mises stress and max principal stress?

Von Mises is a theoretical measure of **stress** used to estimate yield failure criteria in ductile materials and is also popular in fatigue strength calculations (where it is signed positive or negative according to the dominant **Principal stress**), whilst **Principal stress** is a more "real" and directly measurable **stress**

5.) What is plane strain problem?

A **plane strain problem** could be taken as one in which the **strain** in the z-direction is the same at all points in the (x, y) **plane**.

6.) Define Uniquiness.

In mathematics, a **uniqueness theorem** is a **theorem** proving that certain conditions determine a unique solution. Picard – Lindel öf **theorem**, the **uniqueness** of solutions to first-order differential equations. Thompson **uniqueness theorem** in finite group theory.

7.) What is meant by superposition?

The principle of **superposition** states that, when two or more waves of the same type cross at some point, the resultant displacement at that point is equal to the sum of the displacements due to each individual wave.

8.) What is the difference between plane stress and plane strain?

In mathematical term a state of **plane stress** in one in which **stress** along z-direction is ZERO and a **plane strain** condition is one in which **strain** associated along z-direction is ZERO. For physical understanding of the situation let us consider two plates one thick and the other thin.

9.) Define plane strain.

Plane strain A stress condition in linear elastic fracture mechanics in which there is zero **strain** in the direction normal to the axis of applied tensile stress and direction of crack growth. It is achieved in thick plate, along a direction parallel to the plate.

10.) Which type of stress is plane stress?

Plane Stress: If the stress state at a material particle is such that the only non-zero stress components act in one plane only, the particle is said to be in plane stress. The axes are usually chosen such that the **yx** - plane is the plane in which the stresses ac

11.)

What is Mohr's circle of stress?

Mohr's circle, invented by Christian Otto **Mohr**, is a two-dimensional graphical representation of the transformation law for the Cauchy **stress** tensor. ... Karl Culmann was the first to conceive a graphical representation for **stresses** while considering longitudinal and vertical **stresses** in horizontal beams during bending.

12.) What are the 3 principal stresses?

The three principal stresses are conventionally labelled σ_1, σ_2 and σ_3. σ_1 is the maximum (most tensile) principal stress, σ_3 is the minimum (most compressive) principal stress, and σ_2 is the intermediate principal stress..

Module: 7 Torsion of Prismatic Bars

7.2.1 TORSION OF ELLIPTICAL CROSS-SECTION

Let the warping function is given by

$$\psi = Axy \tag{7.15}$$

where A is a constant. This also satisfies the Laplace equation. The boundary condition gives

$$(Ay - y)\frac{dy}{dS} - (Ax + x)\frac{dx}{dS} = 0$$

or $\quad y(A-1)\dfrac{dy}{dS} - x(A+1)\dfrac{dx}{dS} = 0$

i.e., $(A+1)2x\dfrac{dx}{dS} - (A-1)2y\dfrac{dy}{dS} = 0$

or $\quad \dfrac{d}{dS}[(A+1)x^2 - (A-1)y^2] = 0$

Integrating, we get

$(1+A)x^2+(1-A)y^2$ = constant.

This is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

These two are identical if

$$\frac{a^2}{b^2} = \frac{1-A}{1+A}$$

or $A = \dfrac{b^2 - a^2}{b^2 + a^2}$

Therefore, the function given by

$$\psi = \frac{b^2 - a^2}{b^2 + a^2}xy \tag{7.16}$$

represents the warping function for an elliptic cylinder with semi-axes a and b under torsion. The value of polar moment of inertia J is

$$J = \iint (x^2 + y^2 + Ax^2 - Ay^2)dxdy \tag{7.17}$$

$$= (A+1)\int\int x^2 dxdy + (1-A)\int\int y^2 dxdy$$

$$J = (A+1)I_y+(1-A)I_x \tag{7.18}$$

where $I_x = \dfrac{\pi ab^3}{4}$ and $I_y = \dfrac{\pi a^3 b}{4}$

Substituting the above values in (7.18), we obtain

$$J = \frac{\pi a^3 b^3}{a^2+b^2}$$

But $\theta = \dfrac{M_t}{GI_P} = \dfrac{M_t}{GJ}$

Therefore, $M_t = GJ\theta$

$$= G\theta\,\frac{\pi a^3 b^3}{a^2+b^2}$$

or $$\theta = \frac{M_t}{G}\frac{a^2+b^2}{\pi a^3 b^3}$$

The shearing stresses are given by

$$\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y}+x\right)$$

$$= M_t\,\frac{a^2+b^2}{\pi a^3 b^3}\left(\frac{b^2-a^2}{b^2+a^2}+1\right)x$$

or $\tau_{yz} = \dfrac{2M_t x}{\pi a^3 b}$

Similarly, $\tau_{xz} = \dfrac{2M_t y}{\pi ab^3}$

Therefore, the resultant shearing stress at any point (x, y) is

$$\tau = \sqrt{\tau_{yz}^2+\tau_{xz}^2} = \frac{2M_t}{\pi a^3 b^3}\left[b^4x^2+a^4y^2\right]^{\frac{1}{2}} \tag{7.19}$$

Determination of Maximum Shear Stress

To determine where the maximum shear stress occurs, substitute for x^2 from

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,$$

or $x^2 = a^2(1-y^2/b^2)$

and $\tau = \frac{2M_t}{\pi a^3 b^3}\left[a^2 b^4 + a^2(a^2 - b^2)y^2\right]^{\frac{1}{2}}$

Since all terms under the radical (power 1/2) are positive, the maximum shear stress occurs when y is maximum, i.e., when $y = b$. Thus, maximum shear stress τ_{max} occurs at the ends of the minor axis and its value is

$$\tau_{max} = \frac{2M_t}{\pi a^3 b^3}(a^4 b^2)^{1/2}$$

Therefore, $\tau_{max} = \frac{2M_t}{\pi a b^2}$ (7.20)

For $a = b$, this formula coincides with the well-known formula for circular cross-section. Knowing the warping function, the displacement w can be easily determined.

Therefore, $w = \theta\psi = \frac{M_t(b^2 - a^2)}{\pi a^3 b^3 G}xy$ (7.21)

The contour lines giving w = constant are the hyperbolas shown in the Figure 7.4 having the principal axes of the ellipse as asymptotes.

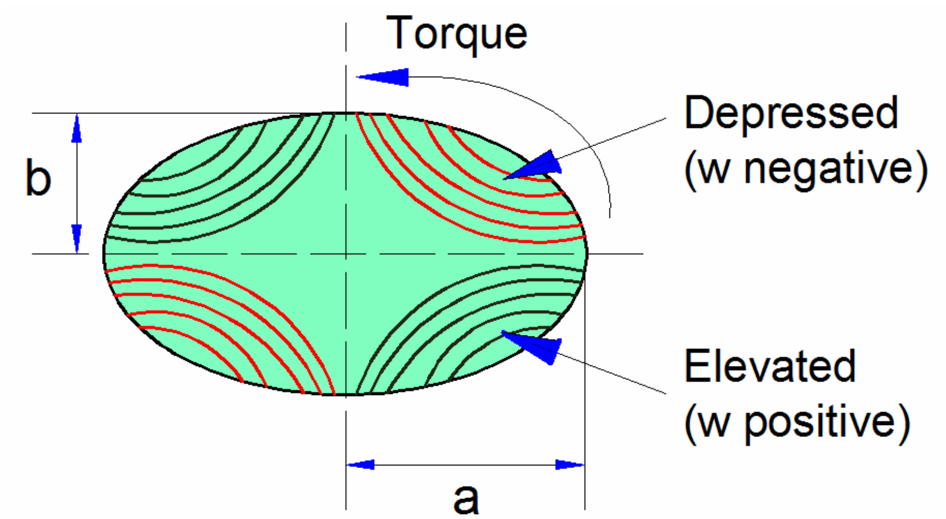


Figure 7.4 Cross-section of elliptic bar and contour lines of *w*

7.2.2 PRANDTL'S MEMBRANE ANALOGY

It becomes evident that for bars with more complicated cross-sectional shapes, more analytical solutions are involved and hence become difficult. In such situations, it is desirable to use other techniques – experimental or otherwise. The membrane analogy introduced by Prandtl has proved very valuable in this regard.

Let a thin homogeneous membrane, like a thin rubber sheet be stretched with uniform tension fixed at it's edge which is a given curve (the cross-section of the shaft) in the xy-plane as shown in the figure 7.5.

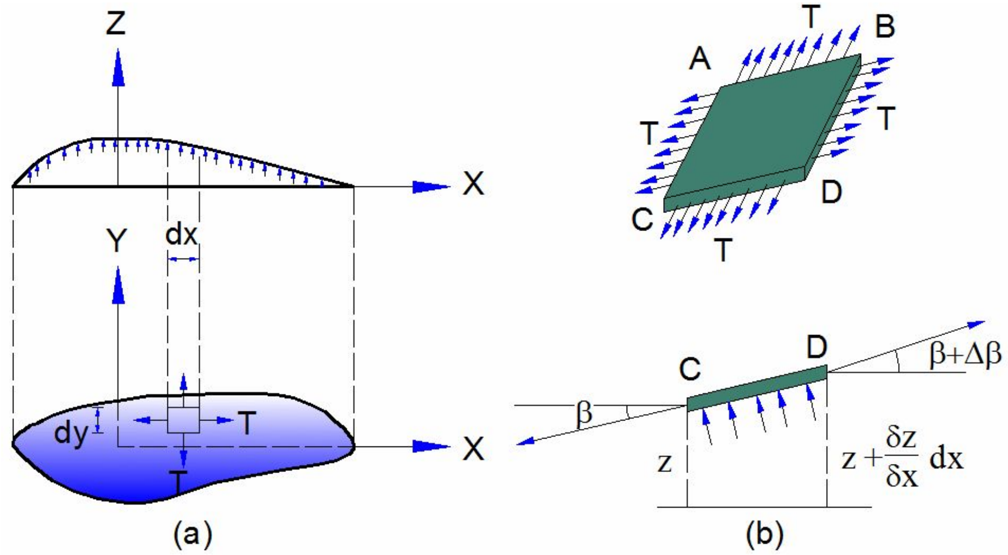


Figure 7.5 Stretching of a membrane

When the membrane is subjected to a uniform lateral pressure p, it undergoes a small displacement z where z is a function of x and y.

Consider the equilibrium of an infinitesimal element ABCD of the membrane after deformation. Let F be the uniform tension per unit length of the membrane. The value of the initial tension F is large enough to ignore its change when the membrane is blown up by the small pressure p. On the face AD, the force acting is $F.dy$. This is inclined at an angle β to the x-axis. Also, tan β is the slope of the face AB and is equal to $\frac{\partial z}{\partial x}$. Hence the component of Fdy in z-direction is $\left(-Fdy\frac{\partial z}{\partial x}\right)$. The force on face BC is also Fdy but is inclined at an angle $(\beta + \Delta\beta)$ to the x-axis. Its slope is, therefore,

$$\frac{\partial z}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)dx$$

and the component of the force in the z-direction is

$$Fdy\left[\frac{\partial z}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)dx\right]$$

Similarly, the components of the forces Fdx acting on face AB and CD are

$-Fdx\frac{\partial z}{\partial y}$ and $Fdx\left[\frac{\partial z}{\partial y}+\frac{\partial}{\partial y}(\frac{\partial z}{\partial y})dy\right]$

Therefore, the resultant force in z-direction due to tension F

$$= -Fdy\frac{\partial z}{\partial x}+Fdy\left[\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x^2}dx\right]-Fdx\frac{\partial z}{\partial y}+Fdx\left[\frac{\partial z}{\partial y}+\frac{\partial^2 z}{\partial y^2}dy\right]$$

$$= F\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)dxdy$$

But the force p acting upward on the membrane element ABCD is $p\ dxdy$, assuming that the membrane deflection is small.

Hence, for equilibrium,

$$F\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right) = -p$$

or $$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = -p/F \quad (7.22)$$

Now, if the membrane tension F or the air pressure p is adjusted in such a way that p/F becomes numerically equal to $2G\theta$, then Equation (7.22) of the membrane becomes identical to Equation (7.8) of the torsion stress function ϕ. Further if the membrane height z remains zero at the boundary contour of the section, then the height z of the membrane becomes numerically equal to the torsion stress function $\phi = 0$. The slopes of the membrane are then equal to the shear stresses and these are in a direction perpendicular to that of the slope.

Further, the twisting moment is numerically equivalent to twice the volume under the membrane [Equation (7.14)].

Table 7.1 Analogy between Torsion and Membrane Problems

Membrane problem	Torsion Problem
Z	ϕ
$\frac{1}{S}$	G
P	2θ
$-\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$	τ_{zy},τ_{zx}
2 (volume beneath membrane)	M_t

The membrane analogy provides a useful experimental technique. It also serves as the basis for obtaining approximate analytical solutions for bars of narrow cross-section as well as for member of open thin walled section.

7.2.3 TORSION OF THIN-WALLED SECTIONS

Consider a thin-walled tube subjected to torsion. The thickness of the tube may not be uniform as shown in the Figure 7.6.

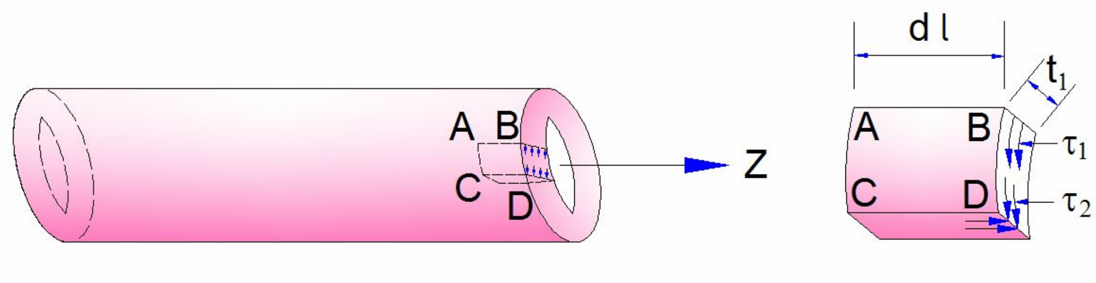


Figure 7.6 Torsion of thin walled sections

Since the thickness is small and the boundaries are free, the shear stresses will be essentially parallel to the boundary. Let τ be the magnitude of shear stress and t is the thickness.

Now, consider the equilibrium of an element of length Δl as shown in Figure 7.6. The areas of cut faces *AB* and *CD* are $t_1 \Delta l$ and $t_2 \Delta l$ respectively. The shear stresses (complementary shears) are τ_1 and τ_2.

For equilibrium in *z*-direction, we have

$$-\tau_1 t_1 \Delta l + \tau_2 t_2 \Delta l = 0$$

Therefore, $\tau_1 t_1 = \tau_2 t_2 = q$ = constant

Hence the quantity τt is constant. This is called the shear flow q, since the equation is similar to the flow of an incompressible liquid in a tube of varying area.

Determination of Torque Due to Shear and Rotation

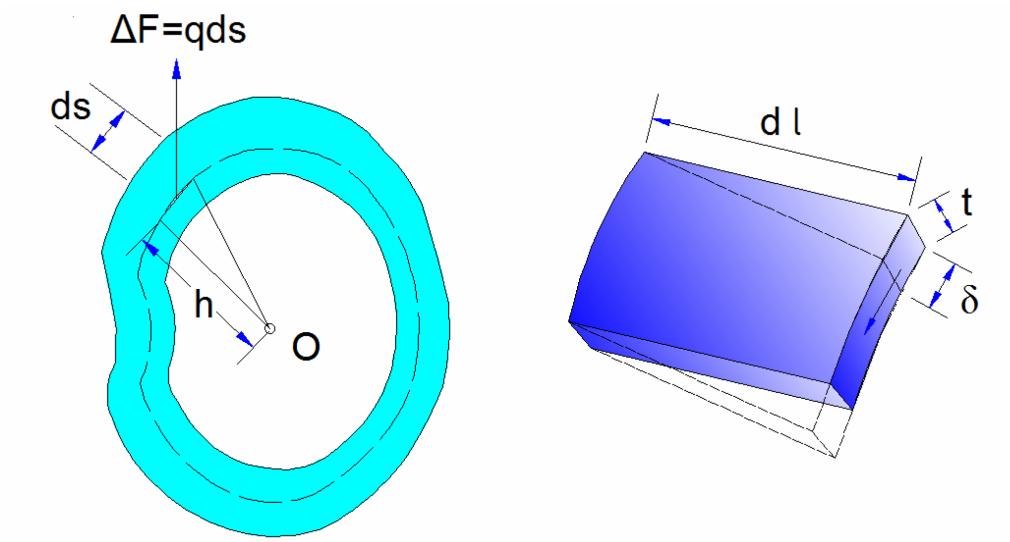


Figure 7.7 Cross section of a thin-walled tube and torque due to shear

Consider the torque of the shear about point O (Figure 7.7).
The force acting on the elementary length dS of the tube = $\Delta F = \tau t\, dS = q\, dS$
The moment arm about O is h and hence the torque = $\Delta M_t = (qdS)\, h$
Therefore, $\Delta M_t = 2qdA$

where dA is the area of the triangle enclosed at O by the base dS.

Hence the total torque is

$M_t = \Sigma\, 2qdA+$

Therefore, $M_t = 2qA$ (7.23)

where A is the area enclosed by the centre line of the tube. Equation (7.23) is generally known as the "Bredt-Batho" formula.

To Determine the Twist of the Tube

In order to determine the twist of the tube, Castigliano's theorem is used. Referring to Figure 7.7(b), the shear force on the element is $\tau\, t\, dS = qdS$. Due to shear strain γ, the force does work equal to ΔU

i.e., $\Delta U = \frac{1}{2}(\tau\, tdS)\delta$

$$= \frac{1}{2}(\tau\, tdS)\gamma.\Delta l$$

$$= \frac{1}{2}(\tau\, tdS).\Delta l.\frac{\tau}{G} \text{ (since } \tau = G\gamma \text{)}$$

$$= \frac{\tau^2 t^2 dS\Delta l}{2Gt}$$

$$= \frac{q^2 dS\Delta l}{2Gt}$$

$$= \frac{q^2 \Delta l}{2G}.\frac{dS}{t}$$

$$\Delta U = \frac{M_t^2 \Delta l}{8A^2 G}.\frac{dS}{t}$$

Therefore, the total elastic strain energy is

$$U = \frac{M_t^2 \Delta l}{8A^2 G}\oint\frac{dS}{t}$$

Hence, the twist or the rotation per unit length (Δl = 1) is

$$\theta = \frac{\partial U}{\partial M_t} = \frac{M_t}{4A^2 G}\oint\frac{dS}{t}$$

or $\theta = \frac{2qA}{4A^2 G}\oint\frac{dS}{t}$

or $\theta = \frac{q}{2AG}\oint\frac{dS}{t}$ (7.24)

7.2.4 TORSION OF THIN-WALLED MULTIPLE-CELL CLOSED SECTIONS

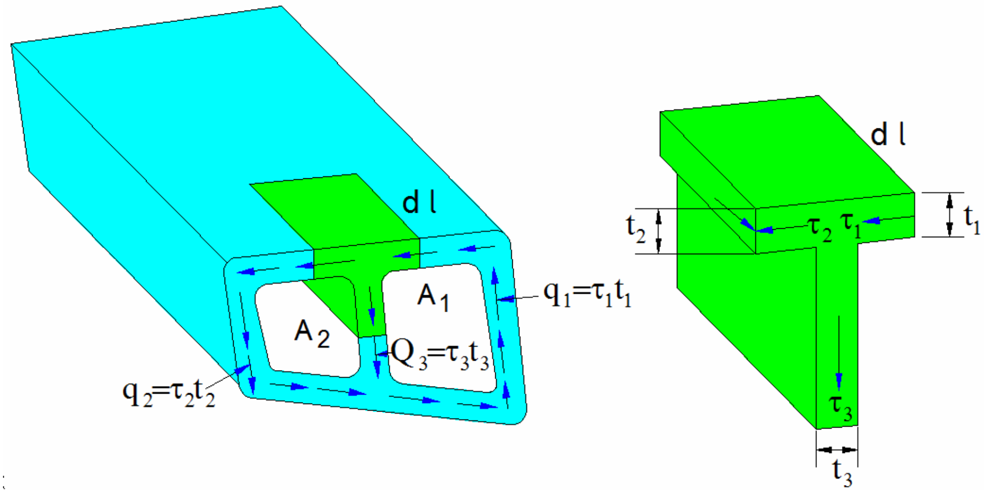


Figure 7.8 Torsion of thin-walled multiple cell closed section

Consider the two-cell section shown in the Figure 7.8. Let A_1 and A_2 be the areas of the cells 1 and 2 respectively. Consider the equilibrium of an element at the junction as shown in the Figure 7.8(b). In the direction of the axis of the tube, we can write

$$-\tau_1 t_1 \Delta l + \tau_2 t_2 \Delta l + \tau_3 t_3 \Delta l = 0$$

or $\tau_1 t_1 = \tau_2 t_2 + \tau_3 t_3$

i.e., $q_1 = q_2 + q_3$

This is again equivalent to a fluid flow dividing itself into two streams. Now, choose moment axis, such as point O as shown in the Figure 7.9.

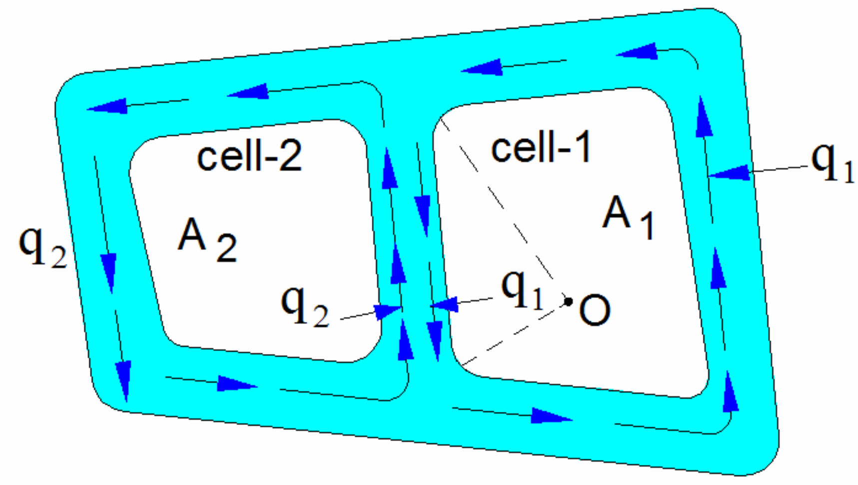


Figure. 7.9 Section of a thin walled multiple cell beam and moment axis

The shear flow in the web is considered to be made of q_1 and $-q_2$, since $q_3 = q_1 - q_2$.
Moment about O due to q_1 flowing in cell 1 (including web) is

$M_{t_1} = 2q_1A_1$

Similarly, the moment about O due to q_2 flowing in cell 2 (including web) is

$M_{t_2} = 2q_2\,(A_2+A_1) - 2q_2A_1$

The second term with the negative sign on the right hand side is the moment due to shear flow q_2 in the middle web.

Therefore, The total torque is

$M_t = M_{t_1} + M_{t_2}$

$$M_t = 2q_1A_1 + 2q_2A_2 \qquad \text{(a)}$$

To Find the Twist (θ)

For continuity, the twist of each cell should be the same.

We have

$$\theta = \frac{q}{2AG}\oint\frac{dS}{t}$$

or $\quad 2G\theta = \dfrac{1}{A}\int\dfrac{qdS}{t}$

Let $a_1 = \oint \frac{dS}{t}$ for Cell 1 including the web

$a_2 = \oint \frac{dS}{t}$ for Cell 2 including the web

$a_{12} = \oint \frac{dS}{t}$ for the web only

Then for Cell 1

$$2G\theta = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2) \tag{b}$$

For Cell 2

$$2G\theta = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1) \tag{c}$$

Equations (a), (b) and (c) are sufficient to solve for q_1, q_2 and θ.

7.2.5 NUMERICAL EXAMPLES

Example 7.1

A hollow aluminum tube of rectangular cross-section shown in Figure below, is subjected to a torque of 56,500 *m-N* along its longitudinal axis. Determine the shearing stresses and the angle of twist. Assume $G = 27.6 \times 10^9$ N/m^2.

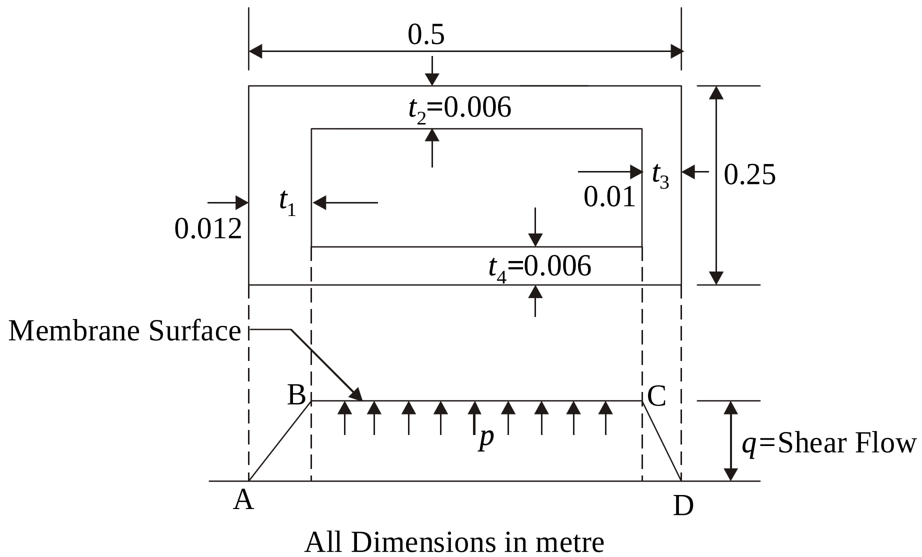


Figure 7.10

Solution: The above figure shows the membrane surface *ABCD*

Now, the Applied torque $= M_t = 2qA$

$$56{,}500 = 2q(0.5\text{x}0.25)$$

$$56{,}500 = 0.25q$$

hence, $q = 226000$ *N/m*.

Now, the shearing stresses are

$$\tau_1 = \frac{q}{t_1} = \frac{226000}{0.012} = 18.833 \times 10^6 N/m^2$$

$$\tau_2 = \frac{q}{t_2} = \frac{226000}{0.006} = 37.667 \times 10^6 N/m^2$$

$$\tau_3 = \frac{226000}{0.01} = 22.6 \times 10^6 N/m^2$$

Now, the angle of twist per unit length is

$$\theta = \frac{q}{2GA}\oint\frac{ds}{t}$$

Therefore,

$$\theta = \frac{226000}{2\text{x}27.6\text{x}10^9\text{x}0.125}\left[\frac{0.25}{0.012} + \frac{0.5}{0.006}(2) + \frac{0.25}{0.01}\right]$$

or $\theta = 0.00696014$ *rad/m*

Example 7.2

The figure below shows a two-cell tubular section as formed by a conventional airfoil shape, and having one interior web. An external torque of 10,000 *Nm* is acting in a clockwise direction. Determine the internal shear flow distribution. The cell areas are as follows:

$A_1 = 680\ cm^2$ $\quad\quad$ $A_2 = 2000\ cm^2$

The peripheral lengths are indicated in Figure

***Solution*:**

For Cell 1, $a_1 = \oint\frac{dS}{t}$ (including the web)

$$= \frac{67}{0.06} + \frac{33}{0.09}$$

therefore, $a_1 = 148.3$

For Cell 2,

$$a_2 = \frac{33}{0.09} + \frac{63}{0.09} + \frac{48}{0.09} + \frac{67}{0.08}$$

Therefore, $a_2 = 2409$

For web,

$$a_{12} = \frac{33}{0.09} = 366$$

Now, for Cell 1,

$$2G\theta = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2)$$

$$= \frac{1}{680}(1483 q_1 - 366 q_2)$$

Therefore, $2G\theta = 2.189 q_1 - 0.54 q_2$ (i)

For Cell 2,

$$2G\theta = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1)$$

$$= \frac{1}{2000}(2409 q_2 - 366 q_1)$$

Therefore, $2G\theta = 1.20 q_2 - 0.18 q_1$ (ii)

Equating (i) and (ii), we get

$$2.18\, q_1 - 0.54 q_2 = 1.20 q_2 - 0.18 q_1$$

or $2.36 q_1 - 1.74 q_2 = 0$

or $q_2 = 1.36 q_1$

The torque due to shear flows should be equal to the applied torque

Hence, from Equation (a),

$M_t = 2q_1 A_1 + 2q_2 A_2$

$10{,}000 \times 100 = 2q_1 \text{ x } 680 + 2q_2 \text{ x } 2000$

$= 1360 q_1 + 4000 q_2$

Substituting for q_2, we get

$10000 \times 100 = 1360 q_1 + 4000 \times 1.36 q_1$

Therefore,
q_1 = 147 *N* and q_2 = 200 *N*

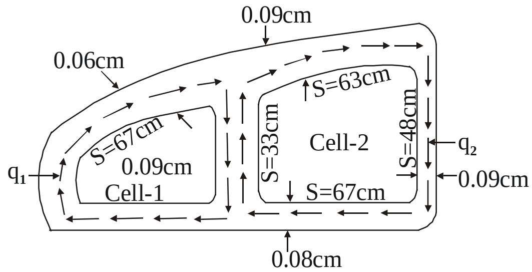


Figure 7.11

Example 7.3
A thin walled steel section shown in figure is subjected to a twisting moment *T*. Calculate the shear stresses in the walls and the angle of twist per unit length of the box.

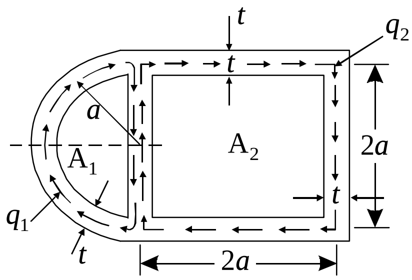


Figure 7.12

Solution: Let A_1 and A_2 be the areas of the cells (1) and (2) respectively.

$$\therefore A_1 = \frac{\pi a^2}{2}$$

$$A_2 = (2a \times 2a) = 4a^2$$

For Cell (1),

$$a_1 = \oint \frac{ds}{t} \text{ (Including the web)}$$

$$a_1 = \left(\frac{\pi a + 2a}{t}\right)$$

For Cell (2),

$$a_2 = \oint \frac{ds}{t}$$

$$= \frac{2a}{t} + \frac{2a}{t} + \frac{2a}{t} + \frac{2a}{t}$$

$$\therefore a_2 = \left(\frac{8a}{t}\right)$$

For web,

$$a_{12} = \left(\frac{2a}{t}\right)$$

Now,
For Cell (1),

$$2G\theta = \frac{1}{A_1}(a_1q_1 - a_{12}q_2)$$

$$= \frac{2}{\pi a^2}\left[\frac{(\pi a + 2a)}{t}q_1 - \left(\frac{2a}{t}\right)q_2\right]$$

$$= \frac{2a}{\pi ta^2}[(2+\pi)q_1 - 2q_2]$$

$$\therefore 2G\theta = \frac{2}{\pi at}[(\pi+2)q_1 - 2q_2] \qquad (1)$$

For Cell (2),

$$2G\theta = \frac{1}{A_2}(a_2q_2 - a_{12}q_1)$$

$$= \frac{1}{4a^2}\left[\frac{8a}{t}q_2 - \frac{2a}{t}q_1\right]$$

$$= \frac{2a}{4a^2t}[4q_2 - q_1]$$

$$\therefore 2G\theta = \frac{1}{2at}[4q_2 - q_1] \qquad (2)$$

Equating (1) and (2), we get,

$$\frac{2}{\pi at}[(\pi+2)q_1 - 2q_2] = \frac{1}{2at}[4q_2 - q_1]$$

or $\frac{2}{\pi}[(\pi+2)q_1 - 2q_2] = \frac{1}{2}[4q_2 - q_1]$

$$\frac{4}{\pi}\left[(\pi+2)q_1 - 2q_2\right] = \left[4q_2 - q_1\right]$$

$$\therefore \frac{4(\pi+2)}{\pi}q_1 - \frac{8}{\pi}q_2 - 4q_2 + q_1 = 0$$

$$\left[\frac{4(\pi+2)}{\pi} + 1\right]q_1 - \left[\frac{8}{\pi} + 4\right]q_2 = 0$$

$$\left[\frac{4(\pi+2)+\pi}{\pi}\right]q_1 - \left[\frac{8+4\pi}{\pi}\right]q_2 = 0$$

or $(4\pi + 8 + \pi)q_1 = (8 + 4\pi)q_2$

$$\therefore q_2 = \left(\frac{5\pi+8}{4\pi+8}\right)q_1$$

But the torque due to shear flows should be equal to the applied torque.

i.e., $T = 2q_1A_1 + 2q_2A_2$ (3)

Substituting the values of q_2, A_1 and A_2 in (3), we get,

$$T = 2q_1\left(\frac{\pi a^2}{2}\right) + 2\left(\frac{5\pi+8}{4\pi+8}\right)q_1.4a^2$$

$$= \pi a^2 q_1 + 8a^2\left(\frac{5\pi+8}{4\pi+8}\right)q_1$$

$$\therefore T = \left[\frac{a^2(\pi^2 + 12\pi + 16)}{(\pi+2)}\right]q_1$$

$$\therefore q_1 = \frac{(\pi+2)T}{a^2(\pi^2 + 12\pi + 16)}$$

Now, from equation (1), we have,

$$2G\theta = \frac{2}{\pi a t}\left[(\pi+2)\frac{(\pi+2)T}{a^2(\pi^2+12\pi+16)} - 2\left(\frac{5\pi+8}{4\pi+8}\right)\frac{(\pi+2)T}{a^2(\pi^2+12\pi+16)}\right]$$

Simplifying, we get the twist as $\theta = \left[\frac{(2\pi+3)T}{2Ga^3t(\pi^2+12\pi+16)}\right]$

Example 7.4

A thin walled box section having dimensions $2a \times a \times t$ is to be compared with a solid circular section of diameter as shown in the figure. Determine the thickness *t* so that the two sections have:

(a) Same maximum shear stress for the same torque.
(b) The same stiffness.

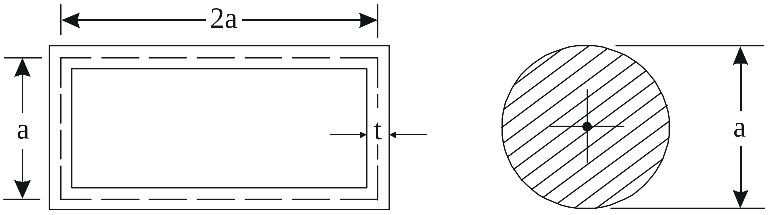


Figure 7.13

Solution: (a) For the box section, we have

$$T = 2qA$$

$$= 2.\tau.t.A$$

$$T = 2.\tau.t.2a \times a$$

$$\therefore \tau = \frac{T}{4a^2 t} \qquad (a)$$

Now, For solid circular section, we have

$$\frac{T}{I_p} = \frac{\tau}{r}$$

Where I_p = Polar moment of inertia

$$\therefore \frac{T}{\left(\frac{\pi a^4}{32}\right)} = \frac{\tau}{\left(\frac{a}{2}\right)}$$

$$or \frac{32T}{\pi a^4} = \frac{2\tau}{a}$$

$$\therefore \tau = \left(\frac{16T}{\pi a^3}\right) \qquad (b)$$

Equating (a) and (b), we get

$$\frac{T}{4a^2 t} = \frac{16T}{\pi a^3} \quad \therefore 64a^2 tT = \pi a^3 T$$

$$\therefore t = \frac{\pi a}{64}$$

(b) The stiffness of the box section is given by

$$\theta = \frac{q}{2GA}\oint \frac{ds}{t}$$

Here $T = 2qA$ $\quad \therefore q = \frac{T}{2A}$

$$\therefore \theta = \frac{T}{4GA^2}\left[\frac{a}{t} + \frac{2a}{t} + \frac{a}{t} + \frac{2a}{t}\right]$$

$$= \frac{6aT}{4GA^2 t}$$

$$= \frac{6aT}{4G\left(2a^2\right)^2 t}$$

$$\therefore \theta = \frac{6aT}{16a^4 Gt} \qquad (c)$$

The stiffness of the Solid Circular Section is

$$\theta = \frac{T}{GI_p} = \frac{T}{G\left(\frac{\pi a^4}{32}\right)} = \frac{32T}{G\pi a^4} \qquad (d)$$

Equating (c) and (d), we get

$$\frac{6aT}{16a^4 Gt} = \frac{32T}{G\pi a^4}$$

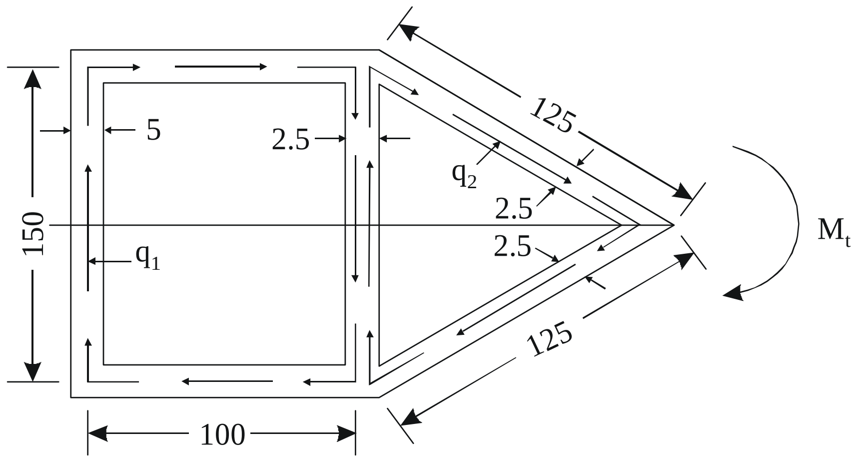
$$\frac{6a}{16t} = \frac{32}{\pi}$$

$$\therefore t = \frac{6\pi a}{16 \times 32}$$

$$\therefore t = \frac{3}{4}\left(\frac{\pi a}{64}\right)$$

Example 7.5

A two-cell tube as shown in the figure is subjected to a torque of 10*kN-m*. Determine the Shear Stress in each part and angle of twist per metre length. Take modulus of rigidity of the material as 83 *kN/mm*^{2}.



All dimensions in mm

Figure 7.14

***Solution*: For Cell 1**

Area of the Cell = $A_1 = 150 \times 100 = 15000 mm^2$

$$a_1 = \oint \frac{ds}{t} \text{ (including web)}$$

$$= \frac{150}{5} + \frac{100}{5} + \frac{150}{2.5} + \frac{100}{5}$$

$$= 130$$

For Cell 2

Area of the cell = $A_2 = \frac{1}{2} \times 150 \times \sqrt{(125)^2 - (75)^2}$

$= 7500 mm^2$

$$\therefore a_2 = \oint \frac{ds}{t} \text{ (including web)}$$

$$= \frac{150}{2.5} + \frac{125}{2.5} + \frac{125}{2.5}$$

$$\therefore a_2 = 160$$

For the web,

$$a_{12} = \frac{150}{2.5} = 60$$

For Cell (1)

$$2G\theta = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2)$$

$$\therefore 2G\theta = \frac{1}{15000}(130q_1 - 60q_2) \qquad (a)$$

For Cell (2)

$$2G\theta = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1)$$

$$= \frac{1}{7500}(160q_2 - 60q_1) \qquad (b)$$

Equating (a) and (b), we get

$$\frac{1}{15000}(130q_1 - 60q_2) = \frac{1}{7500}(160q_2 - 60q_1)$$

Solving, $q_1 = 1.52 q_2$ $\qquad (c)$

Now, the torque due to shear flows should be equal to the applied torque.

i.e., $M_t = 2q_1 A_1 + 2q_2 A_2$

$$10 \times 10^6 = 2q_1(15000) + 2q_2(7500) \qquad (d)$$

Substituting (c) in (d), we get

$$10 \times 10^6 = 2 \times 15000(1.52q_2) + 2q_2(7500)$$

$$\therefore q_2 = 165.02N$$

$$\therefore q_1 = 1.52 \times 165.02 = 250.83N$$

Shear flow in the web = $q_3 = (q_1 - q_2) = (250.83 - 165.02)$

$$\therefore q_3 = 85.81N$$

$$\therefore \tau_1 = \frac{q_1}{t_1} = \frac{250.83}{5} = 50.17N/mm^2$$

$$\tau_2 = \frac{q_2}{t_2} = \frac{165.02}{2.5} = 66.01N/mm^2$$

$$\tau_3 = \frac{q_3}{t_3} = \frac{85.81}{2.5} = 34.32N/mm^2$$

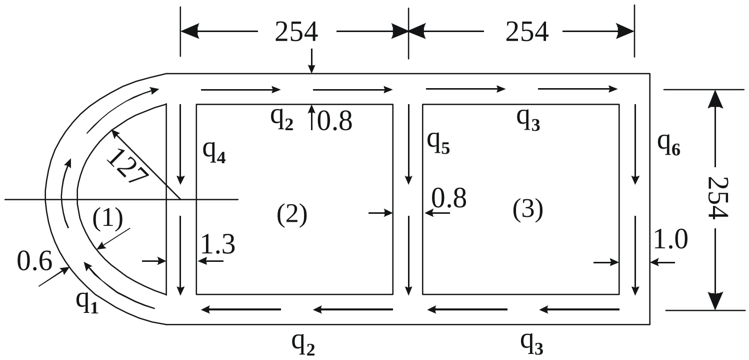
Now, the twist θ is computed by substituting the values of q_1 and q_2 in equation (a) i.e., $2G\theta = \frac{1}{15000}[130 \times 250.83 \times 60 \times 165.02]$

$\therefore \theta = \frac{1}{15000} \times \frac{22706.7}{83 \times 1000} = 1.824 \times 10^{-5}\, radians/mm\, length$

$or \;\; \theta = 1.04$ *degrees/m length*

Example 7.6

A tubular section having three cells as shown in the figure is subjected to a torque of 113 *kN-m*. Determine the shear stresses developed in the walls of the section.



All dimensions in mm

Figure 7.15

Solution: Let $q_1, q_2, q_3, q_4, q_5, q_6$ be the shear flows in the various walls of the tube as shown in the figure. $A_1, A_2, and\;\; A_3$ be the areas of the three cells.

$\therefore A_1 = \frac{\pi}{2}(127)^2 = 25322 mm^2$

$A_2 = 254 \times 254 = 64516 mm^2$

$A_3 = 64516 mm^2$

Now, From the figure,

$q_1 = q_2 + q_4$

$q_2 = q_3 + q_5$

$q_3 = q_6$

or $q_1 = \tau_1 t_1 = \tau_2 t_2 + \tau_4 t_4$

$q_2 = \tau_2 t_2 = \tau_3 t_3 + \tau_5 t_5$ (1)

$q_3 = \tau_3 t_3 = \tau_6 t_6$

Where $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\; and\, \tau_6$ are the Shear Stresses in the various walls of the tube.

Now, The applied torque is

$$M_t = 2A_1q_1 + 2A_2q_2 + 2A_3q_3$$
$$= 2(A_1\tau_1t_1 + A_2\tau_2t_2 + A_3\tau_3t_3)$$

i.e., $113\times10^6 = 2[(25322\tau_1\times0.8)+(64516\tau_2\times0.8)+(64516\times0.8)]$

$\therefore \tau_1 + 3.397(\tau_2+\tau_3) = 3718$ (2)

Now, considering the rotations of the cells and S_1, S_2, S_3, S_4, S_5 *and* S_6 as the length of cell walls,

We have,

$$\tau_1S_1 + \tau_4S_4 = 2G\theta A_1$$
$$-\tau_4S_4 + 2\tau_2S_2 + \tau_5S_5 = 2G\theta A_2 \quad (3)$$
$$-\tau_5S_5 + 2\tau_3S_3 + \tau_6S_6 = 2G\theta A_3$$

Here $S_1 = (\pi\times127) = 398mm$

$S_2 = S_3 = S_4 = S_5 = S_6 = 254mm$

$\therefore$ (3) can be written as

$$398\tau_1 + 254S_4 = 25322G\theta$$
$$-254\tau_2 + 2\times254\times\tau_2 + 254\tau_5 = 64516G\theta \quad (4)$$
$$-254\tau_2 + 2\times254\times\tau_3 + 254\tau_6 = 64516G\theta$$

Now, Solving (1), (2) and (4) we get

$\tau_1 = 40.4N/mm^2$

$\tau_2 = 55.2N/mm^2$

$\tau_3 = 48.9N/mm^2$

$\tau_4 = -12.7N/mm^2$

$\tau_6 = 36.6N/mm^2$

SOLID MECHANICS SHORT QUESTIONS AND ANSWERS UNIT - IV

1.)

Define thick cylinders.

Thick cylinder is **cylinder** whose wall **thickness** is greater than 1/20 times of its internal diameter. ... Thin **cylinder** is **cylinder** whose wall **thickness** is lesser than 1/20 times of its internal diameter.

2.) What is lame's theory?Or Lame's theory

• Assumptions: • The material is homogeneous and isotropic. • Plane sections of the cylinder perpendicular to the longitudinal axis. remain plane under pressure. That is longitudinal strain is the same at all points in the cylinder.

3.) Which ratio decides whether cylinder is thin or thick?

Let t denotes **thickness** and d denotes diameter of the **cylinder**. **If ratio** of t/d is less than 1/20 than the **cylinder is thin cylinder**. And **if ratio** of t/d is greater than 1/20 than **cylinder** is **thick cylinder**

4.) What are thick cylinders?

Thick cylinder is **cylinder** whose wall **thickness** is greater than 1/20 times of its internal diameter. ... Thin **cylinder** is **cylinder** whose wall **thickness** is lesser than 1/20 times of its internal diameter.

5.) What is hoop stress definition?

Hoop stress is the **circumferential** force per unit areas (Psi) in the pipe wall due to internal pressure. It can be explained as the largest tensile **stress** in a supported pipe carrying a fluid under pressure.

6.) What is the difference between thick and thin?

Density is the main **difference between thick and thin** hair. **Thick** hair has a higher density, **thin** hair's density is lower. ... Those with more than 2,200 strands have **thicker** hair, those with less have **thinner** hair.

7.) What is radial stress in thick cylinder?

The **radial stress** for a **thick**-walled **cylinder** is equal and opposite to the gauge pressure on the inside surface, and zero on the outside surface. The circumferential **stress** and longitudinal **stresses** are usually much larger for pressure vessels, and so for thin-walled instances, **radial stress** is usually neglected.

8.) What is the difference between hoop stress and longitudinal stress?

Longitudinal stress is the **stress in a** pipe wall, acting along the **longitudinal** axis of the pipe.It is produced by the pressure of the fluid **in the** pipe. It is also called as **Hoop stress**. Radial **stress** is **stress** towards or away from the central axis of a component

9.) What is meant by tangential stress?

Definition of **tangential stress**. : a force acting in a generally horizontal direction especially : a force that produces mountain folding and over thrusting.

10.) What is longitudinal stress in cylinder?

Longitudinal Stress Thin Walled Pressure Vessel: When the vessel has closed ends the internal pressure acts on them to develop a force along the axis of the **cylinder**. This is known as the axial or **longitudinal stress** and is usually less than the hoop **stress**.

11. What is the normal stress?

A **normal stress** is a **stress** that occurs when a member is loaded by an axial force. The value of the **normal** force for any prismatic section is simply the force divided by the cross sectional area. A **normal stress** will occur when a member is placed in tension or compression.

12.) What is longitudinal tension?

elevation and lowering of the larynx.

The active **longitudinal tension** of the vocal folds is achieved through the contraction of the vocalis muscle, whereas the passive **longitudinal tension** is achieved through contraction of the cricothyroid muscle.

What is a tangential relationship?

tangential. **Tangential** describes something that's not part of the whole. If you make a comment that is **tangential** to the story you're telling, it's a digression. The story could still be understood without it. In geometry, a tangent is a line that touches a curve in one spot but doesn't intersect it anywhere else.

13.) What is meant by tangential force?

Tangential force. (Mech.) a **force** which acts on a moving body in the direction of a **tangent** to the path of the body, its effect being to increase or diminish the velocity; -distinguished from a normal **force**, which acts at right angles to the **tangent** and changes the direction of the motion without changing the velocity ..

14.) What is meant by radial stress?

Radial stress is **stress** towards or away from the central axis of a component. The walls of pressure vessels generally undergo tri-axial loading. For cylindrical pressure vessels, the normal loads on a wall element are the longitudinal **stress**, the circumferential (hoop) **stress** and the **radial stress**.

15.) What is meant by circumferential stress?

The **stresses** induced in the cylinder due to the **circumferential** failure is called **circumferential stress**/ **hoop stress**. **Hoop's stress** in thin cylinders. In thin cylinders, the pressure due to the fluid inside causes a bursting force on to the cylinder walls due to which the **stress** are induced in the cylinder.

16.) What is torsional testing?

The purpose of a **torsion test** is to determine the behavior a material or **test** sample exhibits when twisted or under **torsional** forces as a result of applied moments that cause shear stress about the axis.

17.) What are the advantages of hollow shaft over solid shaft?

Hollow shafts are much lighter than **solid shafts** and can transmit same torque like **solid shafts** of the same dimensions. More over less energy is necessary to acceleration and deceleration of **hollow shafts**. Therefore **hollow shafts** have great potential for use in power transmission in automotive industry

18.) What is shear and torsion?

In **shear** force forces are parallel and in opposite direction and causes **shear** force before brakedown. Eg … stress in material while performing **shear** stress test on UTM. In case of **torsion** force acting in tangential direction and causes **twisting** moment.

19.) What is torsional shear stress?

Torsional shear stress or **Torsional stress** is the **shear stress** produced in the shaft due to the twisting. This twisting in the shaft is caused by the couple acting on it.

20.) What is the theory of torsion?

In solid mechanics, **torsion** is the twisting of an object due to an applied torque, therefore is expressed in N. ... The **theory of Torsion** is based on the following Assumptions : The material in the shaft is uniform throughout. The twist along the shaft is uniform. The shaft is of uniform circular cross section throughout.

21.) What is difference between torque and torsion?

Torque and torsion are both related to turning effects experienced by a body. The main **difference between torque and torsion** is that **torque** describes something that is capable of producing an angular acceleration, whereas **torsion** describes the twist formed **in a** body due to a **torque**.

SOLID MECHANICS SHORT QUESTIONS AND ANSWERS UNIT - IV

1.)
Define thick cylinders.

Thick cylinder is **cylinder** whose wall **thickness** is greater than 1/20 times of its internal diameter. ... Thin **cylinder** is **cylinder** whose wall **thickness** is lesser than 1/20 times of its internal diameter.

2.) What is lame's theory?Or Lame's theory

• Assumptions: • The material is homogeneous and isotropic. • Plane sections of the cylinder perpendicular to the longitudinal axis. remain plane under pressure. That is longitudinal strain is the same at all points in the cylinder.

3.) Which ratio decides whether cylinder is thin or thick?

Let t denotes **thickness** and d denotes diameter of the **cylinder**. **If ratio** of t/d is less than 1/20 than the **cylinder is thin cylinder**. And **if ratio** of t/d is greater than 1/20 than **cylinder** is **thick cylinder**

4.) What are thick cylinders?

Thick cylinder is **cylinder** whose wall **thickness** is greater than 1/20 times of its internal diameter. ... Thin **cylinder** is **cylinder** whose wall **thickness** is lesser than 1/20 times of its internal diameter.

5.) What is hoop stress definition?

Hoop stress is the **circumferential** force per unit areas (Psi) in the pipe wall due to internal pressure. It can be explained as the largest tensile **stress** in a supported pipe carrying a fluid under pressure.

6.) What is the difference between thick and thin?

Density is the main **difference between thick and thin** hair. **Thick** hair has a higher density, **thin** hair's density is lower. ... Those with more than 2,200 strands have **thicker** hair, those with less have **thinner** hair.

7.) What is radial stress in thick cylinder?

The **radial stress** for a **thick**-walled **cylinder** is equal and opposite to the gauge pressure on the inside surface, and zero on the outside surface. The circumferential **stress** and longitudinal **stresses** are usually much larger for pressure vessels, and so for thin-walled instances, **radial stress** is usually neglected.

8.) What is the difference between hoop stress and longitudinal stress?

Longitudinal stress is the **stress in a** pipe wall, acting along the **longitudinal** axis of the pipe.It is produced by the pressure of the fluid **in the** pipe. It is also called as **Hoop stress**. Radial **stress** is **stress** towards or away from the central axis of a component

9.) What is meant by tangential stress?

Definition of **tangential stress**. : a force acting in a generally horizontal direction especially : a force that produces mountain folding and over thrusting.

10.) What is longitudinal stress in cylinder?

Longitudinal Stress Thin Walled Pressure Vessel: When the vessel has closed ends the internal pressure acts on them to develop a force along the axis of the **cylinder**. This is known as the axial or **longitudinal stress** and is usually less than the hoop **stress**.

11. What is the normal stress?

A **normal stress** is a **stress** that occurs when a member is loaded by an axial force. The value of the **normal** force for any prismatic section is simply the force divided by the cross sectional area. A **normal stress** will occur when a member is placed in tension or compression.

12.) What is longitudinal tension?

elevation and lowering of the larynx.

The active **longitudinal tension** of the vocal folds is achieved through the contraction of the vocalis muscle, whereas the passive **longitudinal tension** is achieved through contraction of the cricothyroid muscle.

What is a tangential relationship?

tangential. **Tangential** describes something that's not part of the whole. If you make a comment that is **tangential** to the story you're telling, it's a digression. The story could still be understood without it. In geometry, a tangent is a line that touches a curve in one spot but doesn't intersect it anywhere else.

13.) What is meant by tangential force?

Tangential force. (Mech.) a **force** which acts on a moving body in the direction of a **tangent** to the path of the body, its effect being to increase or diminish the velocity; -distinguished from a normal **force**, which acts at right angles to the **tangent** and changes the direction of the motion without changing the velocity ..

14.) What is meant by radial stress?

Radial stress is **stress** towards or away from the central axis of a component. The walls of pressure vessels generally undergo tri-axial loading. For cylindrical pressure vessels, the normal loads on a wall element are the longitudinal **stress**, the circumferential (hoop) **stress** and the **radial stress**.

15.) What is meant by circumferential stress?

The **stresses** induced in the cylinder due to the **circumferential** failure is called **circumferential stress/ hoop stress**. **Hoop's stress** in thin cylinders. In thin cylinders, the pressure due to the fluid inside causes a bursting force on to the cylinder walls due to which the **stress** are induced in the cylinder.

16.) What is torsional testing?

The purpose of a **torsion test** is to determine the behavior a material or **test** sample exhibits when twisted or under **torsional** forces as a result of applied moments that cause shear stress about the axis.

17.) What are the advantages of hollow shaft over solid shaft?

Hollow shafts are much lighter than **solid shafts** and can transmit same torque like **solid shafts** of the same dimensions. More over less energy is necessary to acceleration and deceleration of **hollow shafts**. Therefore **hollow shafts** have great potential for use in power transmission in automotive industry

18.) What is shear and torsion?

In **shear** force forces are parallel and in opposite direction and causes **shear** force before brakedown. Eg … stress in material while performing **shear** stress test on UTM. In case of **torsion** force acting in tangential direction and causes **twisting** moment.

19.) What is torsional shear stress?

Torsional shear stress or **Torsional stress** is the **shear stress** produced in the shaft due to the twisting. This twisting in the shaft is caused by the couple acting on it.

20.) What is the theory of torsion?

In solid mechanics, **torsion** is the twisting of an object due to an applied torque, therefore is expressed in N. ... The **theory of Torsion** is based on the following Assumptions : The material in the shaft is uniform throughout. The twist along the shaft is uniform. The shaft is of uniform circular cross section throughout.

21.) What is difference between torque and torsion?

Torque and torsion are both related to turning effects experienced by a body. The main **difference between torque and torsion** is that **torque** describes something that is capable of producing an angular acceleration, whereas **torsion** describes the twist formed **in a** body due to a **torque**.

SOLID MECHANICS SHORT QUESTIONS AND ANSWERS UNIT-V

1.) What is the energy method ?

Rayleigh's **method** is based on the principle of conservation of **energy**. ... The kinetic **energy** is stored in the mass and is proportional to the square of the velocity. The potential **energy** includes strain **energy** that is proportional to elastic deformations and the work done by the applied forces.

2.) What is the difference between elasticity and plasticity?

Elasticity is defined as the property which enables a material to get back to (or recover) its original shape, after the removal of applied force. For example **Plasticity** is defined as the property which enables a material to be deformed continuously and permanently without rupture during the application of force.

3.) What does stress concentration mean?

A **stress concentration** (often called **stress** raisers or **stress** risers) is a location in an object where **stress** is concentrated. ... A material **can** fail, via a propagating crack, when a concentrated **stress** exceeds the material's theoretical cohesive strength.

4.) Define potential energy methods

Potential energy is that **energy** which an object has because of its position. It is called **potential energy** because it has the **potential** to be converted into other forms of **energy**, such as kinetic **energy**.

5.) Define von Mises yield criterion.

The von Mises yield criterion (also known as the maximum distortion energy criterion) suggests that yielding of a ductile material begins when the second deviatoric stress invariant reaches a critical value. It is part of plasticity theory that applies best to ductile materials, such as some metals.

6.) What is the difference between von Mises and Tresca?

Mises is smooth, while **Tresca** has corners. At the crystal level (single grain) yielding does associate with dislocation movement on slip planes. This is caused by shear stress on the slip system (resolved shear stress).

7.) Why von Mises stress is used?

Von Mises stress is a value **used** to determine if a given material will yield or fracture. It is mostly **used** for ductile materials, such as metals.

8.) Is von Mises or Tresca more conservative?

The **Tresca** theory is **more conservative** than the **von Mises** theory. It predicts a narrower elastic region. The **Tresca** criterion can be safer from the design point of view, but it could lead the engineer to take unnecessary measures to prevent an unlikely failure. ... **Von Mises** versus **Tresca** criteria in a 2D system.

9.) What is the difference between von Mises stress and principal stress?

Von Mises is a theoretical measure of **stress** used to estimate yield failure criteria in ductile materials and is also popular in fatigue strength calculations (where it is signed positive or negative according to the dominant **Principal stress**), whilst **Principal stress** is a more "real" and directly measurable **stress**

10.) Define theory of strength.

Definition. In mechanics of materials, the **strength** of a material is its ability to withstand an applied load without failure or plastic deformation. The field of **strength** of materials deals with forces and deformations that result from their acting on a material.

11.) What is Mohr's strength theory of soil?

The **Mohr theory** is virtually an empirical **theory** of yield which accounts for the behavior of permanently deformed materials. As portrayed on a **Mohr** stress diagram the **theory** assumes a functional relation between mean stress and maximum shear stress on the plane of failure.

12.) What are the different theories of failure?

There are five **theories of failure**: Shear strain energy **theory**. Total strain energy **theory**. Maximum shear stress **theory**

13.) What is Rankine theory of failure?

Rankine theory. **Rankine's Theory** assumes that **failure** will occur when the maximum principal stress at any point reaches a value equal to the tensile stress in a simple tension specimen at **failure**. ... **Rankine's theory** is satisfactory for brittle materials, and not applicable to ductile materials.

14.) What is the maximum shear stress theory?

The **Maximum Shear Stress theory** states that failure occurs when the **maximum shear stress** from a combination of principal **stresses** equals or exceeds the value obtained for the **shear stress** at yielding in the uniaxial tensile test.

15.) What is principal stress theory?

Maximum **principle stress theory** or normal **stress theory** says that, yielding occurs at a point in a body, when **principle stress** (maximum normal **stress**) in a biaxial system reaches limiting yield value of that material under simple tension test. ... That's why this **theory** preferred for brittle materials.

16.) What is distortion energy theory?

The **distortion energy theory** is a failure **theory** that is used to predict the failure of a tough material. It is based on the assumption that the proportion of **energy** that causes a component to change shape is a crucial factor in relation to the Material stress. An equivalent stress

Energy and Extremum Principles

STRAIN ENERGY

When an unstressed elastic body is subjected to a system of external loads, when according to the first Law of Thermodynamics

$$W_E + Q = \Delta E \longrightarrow (1)$$

Where

W_E - is the work done by the applied forces during the loading process,

Q - is the heat absorbed by the body from the surroundings

ΔE - is the change of energy associated with the body as a result of the loading.

For an adiabatic deformation, $Q = 0$.

The change in energy ΔE consists of a change in kinetic energy T, plus a change in internal energy, U. If the loads are applied very slowly so that a state of equilibrium is maintained during the entire process, then T = 0 and ΔE represents only a change in the internal energy U,

$$W_E = U \longrightarrow (2)$$

i.e. The mechanical workdone by the applied loads is equal to the change in the internal energy. This stored energy is called strain energy.

Consider an element of volume dv, of a body which is subjected to a single component of stress σ_x. During an ~~increa~~ increment of strain $d\varepsilon_x$, the workdone is equal to the multiplication of the force $\sigma_x . dy . dz$. and extension $d\varepsilon_x . dx$. There fore the energy dU stored in the element when the strain has reached its final value ε_x is

$$dU = \int_0^{\varepsilon_x} \sigma_x . d\varepsilon_x . dx . dy . dz$$

$$= \int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x \, dV \longrightarrow (3)$$

$$U = \int_V \left(\int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x \right) dV \longrightarrow (4)$$

Likewise, if a body is subjected to a generalised state of stress, then

$$U = \int_V \left(\int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x + \int_0^{\varepsilon_y} \sigma_y \, d\varepsilon_y + \int_0^{\varepsilon_z} \sigma_z \, d\varepsilon_z + \int_0^{\gamma_{xy}} \tau_{xy} . d\gamma_{xy} + \int_0^{\gamma_{yz}} \tau_{yz} \, d\gamma_{yz} + \int_0^{\gamma_{xz}} \tau_{xz} . d\gamma_{xz} \right) dV.$$

$$U = \int_V \left(\int_0^{\varepsilon_{ij}} \sigma_{ij} \, d\varepsilon_{ij} \right) dV. \quad \rightarrow (5)$$

[illegible]

[illegible]
[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]
[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]
[illegible]

Now for an isotropic linear elastic material

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda \delta_{ij} \cdot \varepsilon_{kk} \rightarrow (6)$$

Sub in eq (5) we get

$$U = \int_V \left(G\varepsilon_{ij} \cdot \varepsilon_{ij} + \frac{\lambda}{2} \varepsilon_{kk}^2 \right) dV \rightarrow (7)$$

$$U = \int_V \left(\frac{1}{4G} \sigma_{ij}\sigma_{ij} - \frac{\lambda}{4G(3G+3\lambda)} \sigma_{kk}^2 \right) dV \rightarrow (8)$$

In the expanded form these equations may be writtern as

$$U = \int_V \left[\frac{E\mu}{2(1+\mu)(1-2\mu)} (\epsilon_x + \epsilon_y + \epsilon_z)^2 + G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{G}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \right] dV \rightarrow (9)$$

and

$$U = \int_V \left[\frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\mu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right] dV \rightarrow (10)$$

Where the elastic constants E, μ are related to the Lame's Constants G, λ.

Now for an isotropic linear elastic material

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda \delta_{ij} \cdot \epsilon_{kk} \rightarrow (6)$$

Sub in eq (5) we get

$$U = \int_V \left(G\epsilon_{ij} \cdot \epsilon_{ij} + \frac{\lambda}{2} \epsilon_{kk}^2 \right) dV \rightarrow (7)$$

$$U = \int_V \left(\frac{1}{4G} \sigma_{ij}\sigma_{ij} - \frac{\lambda}{4G(3G+3\lambda)} \sigma_{kk}^2 \right) dv \rightarrow (8)$$

In the expanded form these equations may be writtern as

$$U = \int_V \left[\frac{E\mu}{2(1+\mu)(1-2\mu)} (\epsilon_x + \epsilon_y + \epsilon_z)^2 + G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{G}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \right] dV \rightarrow (9)$$

and

$$U = \int_V \left[\frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\mu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right] dV \rightarrow (10)$$

Where the elastic Constants E, μ are related to the Lame's Constants G, λ.

VIRTUAL WORK

The virtual displacement at a point i in the directions of Q, is defined as the infinitesimal virtual distortion δq_i. The virtual work done by a system of forces distributed over the surface of the body is found by multiplying the surface tractions T_i by the corresponding virtual displacements δU_i and integrating over the total surface S: The virtual work of forces distributed throughout the volume is obtained by and integration over the volume V of the product of body forces B_i and the virtual displacements δU_i.

Hence

$$\delta W_E = \int_S T_i \delta U_i ds + \int_V B_i \delta U_i dV \quad (1)$$

$$T_i = \sigma_{ij} n_j$$

$$\int_S T_i \delta U_i ds = \int_S \sigma_{ij} n_j \delta U_i ds = \int_V (\sigma_{ij} \delta U_i)_{,j} dV \quad (2)$$

$$\delta W_E = \int_V \left[(\sigma_{ij,j} + B_i) \delta U_i + \sigma_{ij} \delta U_{i,j} \right] dV \quad (3)$$

VIRTUAL WORK

The virtual displacement at a point i in the directions of Q, is defined as the infinitesimal virtual distortion δq_i. The virtual work done by a system of forces distributed over the surface of the body is found by multiplying the surface tractions T_i by the corresponding virtual displacements δU_i and integrating over the total surface S: The virtual work of forces distributed throughout the volume is obtained by and integration over the volume V of the product of body forces B_i and the virtual displacements δU_i.

Hence

$$\delta W_E = \int_S T_i \delta U_i ds + \int_V B_i \delta U_i dv \quad \to (1)$$

$$T_i = \sigma_{ij} n_j$$

$$\int_S T_i \delta U_i ds = \int_S \sigma_{ij} n_j \delta U_i ds = \int_V (\sigma_{ij} \delta U_i)_{,j} dv \quad \to (2)$$

$$\delta W_E = \int_V \left[(\sigma_{ij,j} + B_i) \delta U_i + \sigma_{ij} \delta U_{i,j} \right] dv \quad \to (3)$$

For the body to be in equilibrium,

$\sigma_{ij,j} + B_i = 0$ Hence

$$\delta W_E = \int_V \sigma_{ij}\, \delta u_{i,j}\, dv \rightarrow (4)$$

Now

$$\delta \epsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$$

$$\sigma_{ij}\, \delta u_{i,j} = \sigma_{ij}\, \delta \epsilon_{ij} \rightarrow (5)$$

Thus,

$$\int_S T_i\, \delta u_i\, ds + \int_V B_i\, \delta u_i\, dv = \int_V \sigma_{ij}\, \delta \epsilon_{ij}\, dv \rightarrow (6)$$

(OR)

$$\delta W_E = \delta U \rightarrow (7)$$

The quantity δU represents the work done by the actual stresses during the virtual distortion, refered to as the internal virtual work.

Eqn (7) represents the principle of virtual work. The principle of virtual work may be stated as follows:

If a body is in equilibrium and remains in equilibrium while it is subjected to a virtual distortion, the external virtual work δW_E done by the external forces acting on the body is equal to the internal virtual work δU done by the internal stresses.

The converse of this principle is also true. i.e

$$\delta W_E = \delta U$$

for an arbitrary virtual distortion, then the body is in equilibrium.

When the body is subjected to a system of a discreate generalised forces Q_i and zero body forces then

$$\delta W_E = \sum_{i=1}^{n} Q_i \delta q_i \rightarrow (8)$$

PRINCIPLE OF MINIMUM POTENTIAL ENERGY

Consider a body in equilibrium whose deformed Configuration in characterised by the displacement field u_i. Now consider a class of arbitrary displacement $\bar{u}_i$ which are Consistent with all Constraints imposed on the body. These arbitrary displacements will, in general differ from the actual displacements by some amont, say δu_i i.e

$$\bar{u}_i = u_i + \delta u_i \rightarrow (1)$$

The variation δu_i is equivalent to the virtual displacement δu_i.

The strain energy U stored in a deformed, isentropic linearly elastic material is given by

$$U = \int_V \left(G \cdot \epsilon_{ij} \cdot \epsilon_{ij} + \frac{\lambda}{2} \epsilon^2_{kk} \right) dV \rightarrow (2)$$

The first variation of U for a variation in the deformation, i.e for a variation in the strains ϵ_{ij} becomes,

$$\delta U = \delta \int_V \left(G \cdot \epsilon_{ij}\, \epsilon_{ij} + \frac{\lambda}{2}\, \epsilon^2_{kk} \right) dV$$

$$= \int_V \left(G \cdot 2 \cdot \epsilon_{ij} \cdot \delta \epsilon_{ij} + \frac{\lambda}{2} \cdot 2 \cdot \epsilon_{kk} \cdot \delta \epsilon_{ij} \right) dV$$

$$= \int_V \left(2 \cdot G \cdot \epsilon_{ij} + \lambda\, \delta_{ij}\, \epsilon_{kk} \right) \delta \epsilon_{ij}\, dV$$

$$= \int_V \sigma_{ij}\, \delta \epsilon_{ij}\, dV \longrightarrow (3)$$

Thus the internal virtual work may be regarded as the first variation of the strain energy U due to variations in the strain components ϵ_{ij}. Eq (3) is also valid for anisotropic and non-linearly elastic materials.

Similarly, the external virtual work.

$$\delta W_E = \int_S T_i\, \delta u_i\, dS + \int_V B_i\, \delta u_i\, dV.$$

May be regarded as the work done by the surface and body forces during a variation δU_i in the displacement field.

Now we shall assume that the body forces and surface forces are derivable from potential function $\phi(U_i)$ and $\psi(U_i)$ respectively.

$$T_i = -\frac{\partial \psi}{\partial U_i}, \quad B_i = -\frac{\partial \phi}{\partial u_i} \rightarrow (4)$$

$$\delta W_E = \int_S -\frac{\partial \psi}{\partial U_i} \delta U_i \, ds + \int_V -\frac{\partial \phi}{\partial v_i} \delta v_i \, dv$$

$$= -\delta \int_S \psi \, ds - \delta \int_V \phi \, dv \rightarrow (5)$$

(OR)

$$\delta W_E = -\delta V_E \rightarrow (6)$$

Where the potential of the external forces V_E is given by.

$$V_E = \int_S \psi \, ds + \int_V \phi \, dV \rightarrow (7)$$

If the surface and body forces are conservative, i.e they are functions of position only and are independent of the position of the body, then

$$\psi = -T_i U_i \quad , \quad \phi = -B_i U_i \longrightarrow (8)$$

Thus

$$V_E = -\int_S T_i U_i \, ds - \int_V B_i U_i \, dV \longrightarrow (9).$$

Therefore for a body which possesses a strain energy U and external potential V_E, the principle of virtual work may be written as,

$$\delta U - \delta W_E = \delta (U + V_E) = 0 \rightarrow (10)$$

$$\delta \pi = 0 \longrightarrow (11)$$

where $\pi = U + V_E \longrightarrow (12).$

is called the total potential energy of the body:

Eq (11) represents the principle of minimum potential energy, which may be expressed as follows:

If the surface and body forces are Conservative, i.e they are functions of position only and are independent of the position of the body, then

$$\psi = -T_i U_i \quad , \quad \phi = -B_i U_i \rightarrow (8)$$

Thus

$$V_E = -\int_S T_i U_i \, ds - \int_V B_i U_i \, dV \rightarrow (9).$$

Therefore for a body which possesses a strain energy U and external potential V_E, the principle of virtual work may be written as,

$$\delta U - \delta W_E = \delta (U + V_E) = 0 \rightarrow (10)$$

$$\delta \pi = 0 \longrightarrow (11)$$

Where $\pi = U + V_E \longrightarrow (12).$

is called the total potential energy of the body:

Eq (11) represents the principle of minimum potential energy, which may be expressed as follows:

Among all the admissible displacements u_i which satisfy the prescribed geometrical boundry Conditions, the actual displacement make the total potential energy of the body is minimum.

The principle of minimum potential energy is applicable to only to elastic bodies linear or non linear, acted upon by forces which are derivable from potential functions.

For an isentropic linearly elastic body subject to a discrete Conservative forces Q_i, we have.

$$V_E = -\sum_{i=1}^{n} Q_i q_i \longrightarrow (13)$$

Energy and Extremum Principles

STRAIN ENERGY

When an unstressed elastic body is subjected to a system of external loads, when according to the first Law of Thermodynamics

$$W_E + Q = \Delta E \longrightarrow (I)$$

Where

W_E - is the work done by the applied forces during the loading process,

Q - is the heat absorbed by the body from the surroundings

ΔE - is the change of energy associated with the body as a result of the loading.

For an adiabatic deformation, $Q = 0$.

The change in energy ΔE consists of a change in kinetic energy T, plus a change in internal energy, U. If the loads are applied very slowly so that a state of equilibrium is maintained during the entire process, then T = 0 and ΔE represents only a change in the internal energy U,

$$W_E = U \longrightarrow (2)$$

i.e. The mechanical workdone by the applied loads is equal to the change in the internal energy. This stored energy is called strain energy.

Consider an element of volume dv, of a body which is subjected to a single component of stress σ_x. During an ~~increast~~ increment of strain $d\varepsilon_x$, the workdone is equal to the multiplication of the force $\sigma_x . dy . dz$. and extension $d\varepsilon_x . dx$. There fore the energy dU stored in the element when the strain has reached its final value ε_x is

$$dU = \int_0^{\varepsilon_x} \sigma_x . d\varepsilon_x . dx . dy . dz$$

$$= \int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x \, dV \longrightarrow (3)$$

$$U = \int_V \left(\int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x \right) dV \longrightarrow (4)$$

Likewise, if a body is subjected to a generalised state of stress, then

$$U = \int_V \left(\int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x + \int_0^{\varepsilon_y} \sigma_y \, d\varepsilon_y + \int_0^{\varepsilon_z} \sigma_z \, d\varepsilon_z + \int_0^{\gamma_{xy}} \tau_{xy} . d\gamma_{xy} + \int_0^{\gamma_{yz}} \tau_{yz} \, d\gamma_{yz} + \int_0^{\gamma_{xz}} \tau_{xz} . d\gamma_{xz} \right) dV.$$

$$U = \int_V \left(\int_0^{\varepsilon_{ij}} \sigma_{ij} \, d\varepsilon_{ij} \right) dV. \quad \to (5)$$

(4)

Now for an isotropic linear elastic material

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda \delta_{ij} \cdot \epsilon_{kk} \quad \to (6)$$

Sub in eq (5) we get

$$U = \int_V \left(G\epsilon_{ij} \cdot \epsilon_{ij} + \frac{\lambda}{2} \epsilon^2_{kk} \right) dV \to (7)$$

$$U = \int_V \left(\frac{1}{4G} \sigma_{ij}\sigma_{ij} - \frac{\lambda}{4G(3G + 3\lambda)} \sigma^2_{kk} \right) dV \quad \to (8)$$

In the expanded form these equations may be writtern as

$$U = \int_V \left[\frac{E\mu}{2(1+\mu)(1-2\mu)} (\epsilon_x + \epsilon_y + \epsilon_z)^2 + G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{G}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \right] dV \to (9)$$

and

$$U = \int_V \left[\frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\mu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{2G}(\tau^2_{xy} + \tau^2_{yz} + \tau^2_{zx}) \right] dV \to (10)$$

Where the elastic constants E, μ are related to the Lame's constants G, λ.